Chapter 13

ANTENNAS

The Ten Commandments of Success

1. Hard Work: Hard work is the best investment a man can make.
2. Study Hard: Knowledge enables a man to work more intelligently and effectively.
3. Have Initiative: Ruts often deepen into graves.
4. Love Your Work: Then you will find pleasure in mastering it.
5. Be Exact: Slipshod methods bring slipshod results.
6. Have the Spirit of Conquest: Thus you can successfully battle and overcome difficulties.
7. Cultivate Personality: Personality is to a man what perfume is to the flower.
8. Help and Share with Others: The real test of business greatness lies in giving opportunity to others.
9. Be Democratic: Unless you feel right toward your fellow men, you can never be a successful leader of men.
10. In all Things Do Your Best: The man who has done his best has done everything. The man who has done less than his best has done nothing.

—CHARLES M. SCHWAB

13.1 INTRODUCTION

Up until now, we have not asked ourselves how EM waves are produced. Recall that electric charges are the sources of EM fields. If the sources are time varying, EM waves propagate away from the sources and radiation is said to have taken place. Radiation may be thought of as the process of transmitting electric energy. The radiation or launching of the waves into space is efficiently accomplished with the aid of conducting or dielectric structures called *antennas.* Theoretically, any structure can radiate EM waves but not all structures can serve as efficient radiation mechanisms.

An antenna may also be viewed as a transducer used in matching the transmission line or waveguide (used in guiding the wave to be launched) to the surrounding medium or vice versa. Figure 13.1 shows how an antenna is used to accomplish a match between the line or guide and the medium. The antenna is needed for two main reasons: efficient radiation and matching wave impedances in order to minimize reflection. The antenna uses voltage and current from the transmission line (or the EM fields from the waveguide) to launch an EM wave into the medium. An antenna may be used for either transmitting or receiving EM energy.

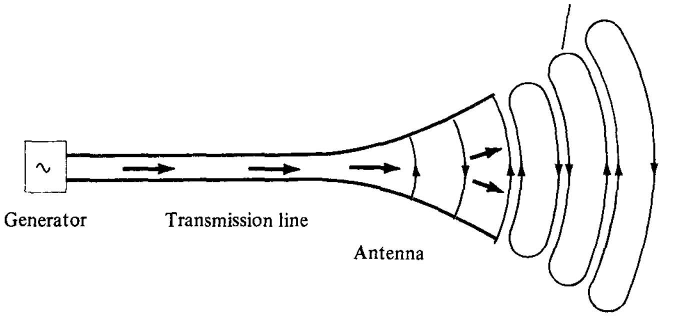

Figure 13.1 Antenna as a matching device between the guiding structure and the surrounding medium.

Typical antennas are illustrated in Figure 13.2. The dipole antenna in Figure 13.2(a) consists of two straight wires lying along the same axis. The loop antenna in Figure 13.2(b) consists of one or more turns of wire. The helical antenna in Figure 13.2(c) consists of a wire in the form of a helix backed by a ground plane. Antennas in Figure 13.2(a–c) are called *wire antennas;* they are used in automobiles, buildings, aircraft, ships, and so on. The horn antenna in Figure 13.2(d), an example of an *aperture antenna,* is a tapered section of waveguide providing a transition between a waveguide and the surroundings. Since it is conveniently flush mounted, it is useful in various applications such as aircraft. The parabolic dish reflector in Figure 13.2(e) utilizes the fact that EM waves are reflected by a conducting sheet. When used as a transmitting antenna, a feed antenna such as a dipole or horn, is placed at the focal point. The radiation from the source is reflected by the dish (acting like a mirror) and a parallel beam results. Parabolic dish antennas are used in communications, radar, and astronomy.

The phenomenon of radiation is rather complicated, so we have intentionally delayed its discussion until this chapter. We will not attempt a broad coverage of antenna theory; our discussion will be limited to the basic types of antennas such as the Hertzian dipole, the half-wave dipole, the quarter-wave monopole, and the small loop. For each of these types, we will determine the radiation fields by taking the following steps:

1. Select an appropriate coordinate system and determine the magnetic vector potential $\mathbf{A}$.
2. Find $\mathbf{H}$ from $\mathbf{B} = \mu\mathbf{H} = \nabla \times \mathbf{A}$.
3. Determine $\mathbf{E}$ from $\nabla \times \mathbf{H} = \varepsilon \dfrac{\partial \mathbf{E}}{\partial t}$ or $\mathbf{E} = \eta\mathbf{H} \times \mathbf{a}_k$ assuming a lossless medium ($\sigma = 0$).
4. Find the far field and determine the time-average power radiated using

$$P_{\text{rad}} = \int \mathcal{P}_{\text{ave}} \cdot d\mathbf{S}, \qquad \text{where} \qquad \mathcal{P}_{\text{ave}} = \frac{1}{2}\,\text{Re}\,(\mathbf{E}_s \times \mathbf{H}_s^*)$$

Note that P_{rad} throughout this chapter is the same as P_{ave} in eq. (10.70).

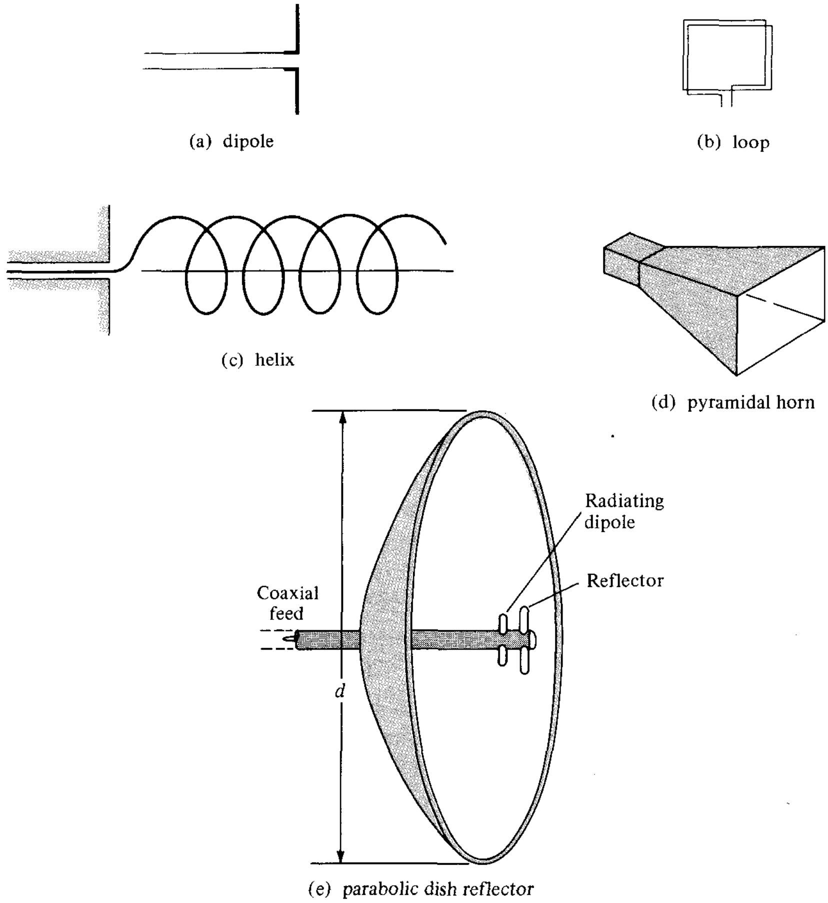

Figure 13.2 Typical antennas.

13.2 HERTZIAN DIPOLE

By a Hertzian dipole, we mean an infinitesimal current element $I\,dl$. Although such a current element does not exist in real life, it serves as a building block from which the field of a practical antenna can be calculated by integration.

Consider the Hertzian dipole shown in Figure 13.3. We assume that it is located at the origin of a coordinate system and that it carries a uniform current (constant throughout the dipole), $I = I_o \cos \omega t$. From eq. (9.54), the retarded magnetic vector potential at the field point P, due to the dipole, is given by

$$\mathbf{A} = \frac{\mu [I]\, dl}{4\pi r} \mathbf{a}_z \tag{13.1}$$

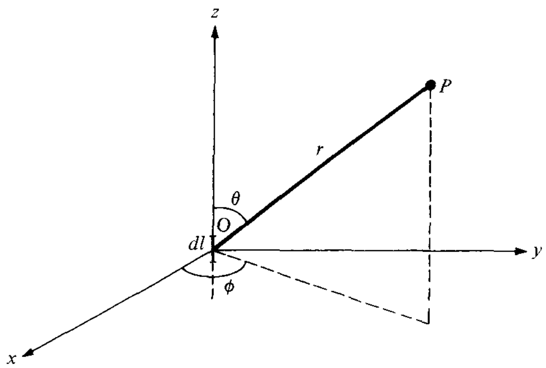

Figure 13.3 A Hertzian dipole carrying current $I = I_o \cos \omega t$.

where $[I]$ is the retarded current given by

$$[I] = I_o \cos \omega \left(t - \frac{r}{u}\right) = I_o \cos (\omega t - \beta r) \qquad (13.2)$$
$$= \text{Re}\,[I_o e^{j(\omega t - \beta r)}]$$

where $\beta = \omega/u = 2\pi/\lambda$, and $u = 1/\sqrt{\mu\varepsilon}$. The current is said to be *retarded* at point P because there is a propagation time delay r/u or phase delay βr from O to P. By substituting eq. (13.2) into eq. (13.1), we may write $\mathbf{A}$ in phasor form as

$$A_{zs} = \frac{\mu I_o dl}{4\pi r} e^{-j\beta r} \qquad (13.3)$$

Transforming this vector in Cartesian to spherical coordinates yields

$$\mathbf{A}_s = (A_{rs}, A_{\theta s}, A_{\phi s})$$

where

$$A_{rs} = A_{zs} \cos \theta, \qquad A_{\theta s} = -A_{zs} \sin \theta, \qquad A_{\phi s} = 0 \qquad (13.4)$$

But $\mathbf{B}_s = \mu \mathbf{H}_s = \nabla \times \mathbf{A}_s$; hence, we obtain the $\mathbf{H}$ field as

$$H_{\phi s} = \frac{I_o dl}{4\pi} \sin \theta \left[\frac{j\beta}{r} + \frac{1}{r^2}\right] e^{-j\beta r} \qquad (13.5a)$$

$$H_{rs} = 0 = H_{\theta s} \qquad (13.5b)$$

We find the $\mathbf{E}$ field using $\nabla \times \mathbf{H} = \varepsilon\, \partial \mathbf{E}/\partial t$ or $\nabla \times \mathbf{H}_s = j\omega\varepsilon \mathbf{E}_s$,

$$E_{rs} = \frac{\eta I_o dl}{2\pi} \cos \theta \left[\frac{1}{r^2} - \frac{j}{\beta r^3}\right] e^{-j\beta r} \qquad (13.6a)$$

$$E_{\theta s} = \frac{\eta I_o dl}{4\pi} \sin \theta \left[\frac{j\beta}{r} + \frac{1}{r^2} - \frac{j}{\beta r^3}\right] e^{-j\beta r} \qquad (13.6b)$$

$$E_{\phi s} = 0 \qquad (13.6c)$$

where

$$\eta = \frac{\beta}{\omega e} = \sqrt{\frac{\mu}{\varepsilon}}$$

A close observation of the field equations in eqs. (13.5) and (13.6) reveals that we have terms varying as $1/r^3$, $1/r^2$, and $1/r$. The $1/r^3$ term is called the *electrostatic field* since it corresponds to the field of an electric dipole [see eq. (4.82)]. This term dominates over other terms in a region very close to the Hertzian dipole. The $1/r^2$ term is called the *inductive field*, and it is predictable from the Biot–Savart law [see eq. 7.3)]. The term is important only at near field, that is, at distances close to the current element. The $1/r$ term is called the *far field or radiation field* because it is the only term that remains at the far zone, that is, at a point very far from the current element. Here, we are mainly concerned with the far field or radiation zone ($\beta r \gg 1$ or $2\pi r \gg \lambda$), where the terms in $1/r^3$ and $1/r^2$ can be neglected in favor of the $1/r$ term. Thus at far field,

$$\boxed{H_{\phi s} = \frac{jI_o \beta dl}{4\pi r} \sin \theta \, e^{-j\beta r}, \qquad E_{\theta s} = \eta \, H_{\phi s}} \tag{13.7a}$$

$$H_{rs} = H_{\theta s} = E_{rs} = E_{\phi s} = 0 \tag{13.7b}$$

Note from eq. (13.7a) that the radiation terms of $H_{\phi s}$ and $E_{\theta s}$ are in time phase and orthogonal just as the fields of a uniform plane wave. Also note that near-zone and far-zone fields are determined respectively to be the inequalities $\beta r \ll 1$ and $\beta r \gg 1$. More specifically, we define the boundary between the near and the far zones by the value of r given by

$$r = \frac{2d^2}{\lambda} \tag{13.8}$$

where d is the largest dimension of the antenna.

The time-average power density is obtained as

$$\begin{aligned} \mathcal{P}_{\text{ave}} &= \frac{1}{2} \text{Re} \, (\mathbf{E}_S \times \mathbf{H}_s^*) = \frac{1}{2} \text{Re} \, (E_{\theta s} H_{\phi s}^* \, \mathbf{a}_r) \\ &= \frac{1}{2} \eta |H_{\phi s}|^2 \, \mathbf{a}_r \end{aligned} \tag{13.9}$$

Substituting eq. (13.7) into eq. (13.9) yields the time-average radiated power as

$$\begin{aligned} P_{\text{rad}} &= \int \mathcal{P}_{\text{ave}} \cdot d\mathbf{S} \\ &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{I_o^2 \eta \beta^2 \, dl^2}{32\pi^2 r^2} \sin^2 \theta \, r^2 \sin \theta \, d\theta \, d\phi \\ &= \frac{I_o^2 \eta \beta^2 \, dl^2}{32\pi^2} 2\pi \int_0^{\pi} \sin^3 \theta \, d\theta \end{aligned} \tag{13.10}$$

But

$$\int_0^{\pi} \sin^3 \theta \, d\theta = \int_0^{\pi} (1 - \cos^2 \theta) \, d(-\cos \theta)$$
$$= \frac{\cos^3 \theta}{3} - \cos \theta \bigg|_0^{\pi} = \frac{4}{3}$$

and $\beta^2 = 4\pi^2/\lambda^2$. Hence eq. (13.10) becomes

$$P_{rad} = \frac{I_o^2 \pi \eta}{3} \left[\frac{dl}{\lambda}\right]^2 \tag{13.11a}$$

If free space is the medium of propagation, $\eta = 120\pi$ and

$$P_{rad} = 40\pi^2 \left[\frac{dl}{\lambda}\right]^2 I_o^2 \tag{13.11b}$$

This power is equivalent to the power dissipated in a fictitious resistance R_{rad} by current $I = I_o \cos \omega t$ that is

$$P_{rad} = I_{rms}^2 R_{rad}$$

or

$$\boxed{P_{rad} = \frac{1}{2} I_o^2 R_{rad}} \tag{13.12}$$

where I_{rms} is the root-mean-square value of I. From eqs. (13.11) and (13.12), we obtain

$$R_{rad} = \frac{2P_{rad}}{I_o^2} \tag{13.13a}$$

or

$$\boxed{R_{rad} = 80\pi^2 \left[\frac{dl}{\lambda}\right]^2} \tag{13.13b}$$

The resistance R_{rad} is a characteristic property of the Hertzian dipole antenna and is called its *radiation resistance.* From eqs. (13.12) and (13.13), we observe that it requires antennas with large radiation resistances to deliver large amounts of power to space. For example, if $dl = \lambda/20$, $R_{rad} \simeq 2\ \Omega$, which is small in that it can deliver relatively small amounts of power. It should be noted that R_{rad} in eq. (13.13b) is for a Hertzian dipole in free space. If the dipole is in a different, lossless medium, $\eta = \sqrt{\mu/\varepsilon}$ is substituted in eq. (13.11a) and R_{rad} is determined using eq. (13.13a).

Note that the Hertzian dipole is assumed to be infinitesimally small ($\beta\, dl \ll 1$ or $dl \leq \lambda/10$). Consequently, its radiation resistance is very small and it is in practice difficult to match it with a real transmission line. We have also assumed that the dipole has a uniform current; this requires that the current be nonzero at the end points of the dipole. This is practically impossible because the surrounding medium is not conducting. However, our analysis will serve as a useful, valid approximation for an antenna with $dl \leq \lambda/10$. A more practical (and perhaps the most important) antenna is the half-wave dipole considered in the next section.

13.3 HALF-WAVE DIPOLE ANTENNA

The half-wave dipole derives its name from the fact that its length is half a wavelength ($\ell = \lambda/2$). As shown in Figure 13.4(a), it consists of a thin wire fed or excited at the mid-point by a voltage source connected to the antenna via a transmission line (e.g., a two-wire line). The field due to the dipole can be easily obtained if we consider it as consisting of a chain of Hertzian dipoles. The magnetic vector potential at P due to a differential length dl (= dz) of the dipole carrying a phasor current $I_s = I_o \cos \beta z$ is

$$dA_{zs} = \frac{\mu I_o \cos \beta z \, dz}{4\pi r'} e^{-j\beta r'} \tag{13.14}$$

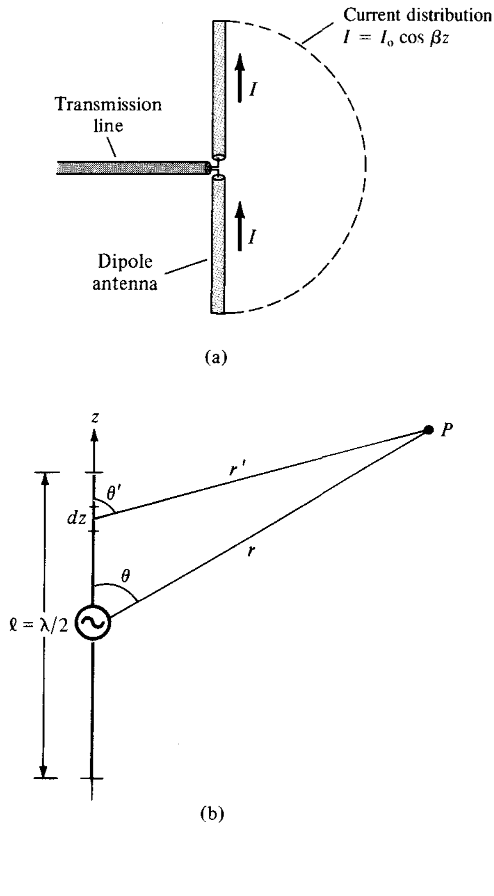

Figure 13.4 A half-wave dipole.

Notice that to obtain eq. (13.14), we have assumed a sinusoidal current distribution because the current must vanish at the ends of the dipole; a triangular current distribution is also possible (see Problem 13.4) but would give less accurate results. The actual current distribution on the antenna is not precisely known. It is determined by solving Maxwell's equations subject to the boundary conditions on the antenna, but the procedure is mathematically complex. However, the sinusoidal current assumption approximates the distribution obtained by solving the boundary-value problem and is commonly used in antenna theory.

If $r \gg \ell$, as explained in Section 4.9 on electric dipoles (see Figure 4.21), then

$$r - r' = z\cos\theta \qquad \text{or} \qquad r' = r - z\cos\theta$$

Thus we may substitute $r' \simeq r$ in the denominator of eq. (13.14) where the magnitude of the distance is needed. For the phase term in the numerator of eq. (13.14), the difference between βr and $\beta r'$ is significant, so we replace r' by $r - z\cos\theta$ and not r. In other words, we maintain the cosine term in the exponent while neglecting it in the denominator because the exponent involves the phase constant while the denominator does not. Thus,

$$\begin{aligned} A_{zs} &= \frac{\mu I_o}{4\pi r}\int_{-\lambda/4}^{\lambda/4} e^{-j\beta(r - z\cos\theta)}\cos\beta z\, dz \\ &= \frac{\mu I_o}{4\pi r}e^{-j\beta r}\int_{-\lambda/4}^{\lambda/4} e^{j\beta z\cos\theta}\cos\beta z\, dz \end{aligned} \tag{13.15}$$

From the integral tables of Appendix A.8,

$$\int e^{az}\cos bz\, dz = \frac{e^{az}(a\cos bz + b\sin bz)}{a^2 + b^2}$$

Applying this to eq. (13.15) gives

$$A_{zs} = \frac{\mu I_o e^{-j\beta r}e^{j\beta z\cos\theta}}{4\pi r}\frac{(j\beta\cos\theta\cos\beta z + \beta\sin\beta z)}{-\beta^2\cos^2\theta + \beta^2}\Bigg|_{-\lambda/4}^{\lambda/4} \tag{13.16}$$

Since $\beta = 2\pi/\lambda$ or $\beta\,\lambda/4 = \pi/2$ and $-\cos^2\theta + 1 = \sin^2\theta$, eq. (13.16) becomes

$$A_{zs} = \frac{\mu I_o e^{-j\beta r}}{4\pi r\beta^2\sin^2\theta}[e^{j(\pi/2)\cos\theta}(0 + \beta) - e^{-j(\pi/2)\cos\theta}(0 - \beta)] \tag{13.17}$$

Using the identity $e^{jx} + e^{-jx} = 2\cos x$, we obtain

$$A_{zs} = \frac{\mu I_o e^{-j\beta r}\cos\left(\dfrac{\pi}{2}\cos\theta\right)}{2\pi r\beta\sin^2\theta} \tag{13.18}$$

We use eq. (13.4) in conjunction with the fact that $\mathbf{B}_s = \mu\mathbf{H}_s = \nabla \times \mathbf{A}_s$ and $\nabla \times \mathbf{H}_s = j\omega\varepsilon\mathbf{E}_s$ to obtain the magnetic and electric fields at far zone (discarding the $1/r^3$ and $1/r^2$ terms) as

$$H_{\phi s} = \frac{jI_o e^{-j\beta r}\cos\left(\dfrac{\pi}{2}\cos\theta\right)}{2\pi r\sin\theta}, \qquad E_{\theta s} = \eta H_{\phi s} \tag{13.19}$$

Notice again that the radiation term of $H_{\phi s}$ and $E_{\theta s}$ are in time phase and orthogonal.

Using eqs. (13.9) and (13.19), we obtain the time-average power density as

$$\begin{aligned}\mathcal{P}_{\text{ave}} &= \frac{1}{2}\eta\,|H_{\phi s}|^2\,\mathbf{a}_r \\ &= \frac{\eta I_o^2\cos^2\left(\dfrac{\pi}{2}\cos\theta\right)}{8\pi^2 r^2\sin^2\theta}\,\mathbf{a}_r\end{aligned} \tag{13.20}$$

The time-average radiated power can be determined as

$$\begin{aligned}P_{\text{rad}} &= \int \mathcal{P}_{\text{ave}} \cdot d\mathbf{S} \\ &= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{\eta I_o^2\cos^2\left(\dfrac{\pi}{2}\cos\theta\right)}{8\pi^2 r^2\sin^2\theta}\, r^2\sin\theta\, d\theta\, d\phi \\ &= \frac{\eta I_o^2}{8\pi^2}\,2\pi\int_0^{\pi}\frac{\cos^2\left(\dfrac{\pi}{2}\cos\theta\right)}{\sin\theta}\,d\theta \\ &= 30\,I_o^2\int_0^{\pi}\frac{\cos^2\left(\dfrac{\pi}{2}\cos\theta\right)}{\sin\theta}\,d\theta\end{aligned} \tag{13.21}$$

where $\eta = 120\pi$ has been substituted assuming free space as the medium of propagation. Due to the nature of the integrand in eq. (13.21),

$$\int_0^{\pi/2}\frac{\cos^2\left(\dfrac{\pi}{2}\cos\theta\right)}{\sin\theta}\,d\theta = \int_{\pi/2}^{\pi}\frac{\cos^2\left(\dfrac{\pi}{2}\cos\theta\right)}{\sin\theta}\,d\theta$$

This is easily illustrated by a rough sketch of the variation of the integrand with θ. Hence

$$P_{\text{rad}} = 60I_o^2\int_0^{\pi/2}\frac{\cos^2\left(\dfrac{\pi}{2}\cos\theta\right)d\theta}{\sin\theta} \tag{13.22}$$

Changing variables, $u = \cos\theta$, and using partial fraction reduces eq. (13.22) to

$$\begin{aligned} P_{\text{rad}} &= 60I_o^2 \int_0^1 \frac{\cos^2 \frac{1}{2}\pi u}{1-u^2}\,du \\ &= 30I_o^2 \left[\int_0^1 \frac{\cos^2 \frac{1}{2}\pi u}{1+u}\,du + \int_0^1 \frac{\cos^2 \frac{1}{2}\pi u}{1-u}\,du \right] \end{aligned} \tag{13.23}$$

Replacing $1 + u$ with v in the first integrand and $1 - u$ with v in the second results in

$$\begin{aligned} P_{\text{rad}} &= 30I_o^2 \left[\int_0^1 \frac{\sin^2 \frac{1}{2}\pi v}{v}\,dv + \int_1^2 \frac{\sin^2 \frac{1}{2}\pi v}{v}\,dv \right] \\ &= 30I_o^2 \int_0^2 \frac{\sin^2 \frac{1}{2}\pi v}{v}\,dv \end{aligned} \tag{13.24}$$

Changing variables, $w = \pi v$, yields

$$\begin{aligned} P_{\text{rad}} &= 30I_o^2 \int_0^{2\pi} \frac{\sin^2 \frac{1}{2} w}{w}\,dw \\ &= 15I_o^2 \int_0^{2\pi} \frac{(1-\cos w)}{w}\,dw \\ &= 15I_o^2 \int_0^{2\pi} \left[\frac{w}{2!} - \frac{w^3}{4!} + \frac{w^5}{6!} - \frac{w^7}{8!} + \cdots \right] dw \end{aligned} \tag{13.25}$$

since $\cos w = 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \frac{w^6}{6!} + \frac{w^8}{8!} - \cdots$. Integrating eq. (13.25) term by term and evaluating at the limit leads to

$$\begin{aligned} P_{\text{rad}} &= 15I_o^2 \left[\frac{(2\pi)^2}{2(2!)} - \frac{(2\pi)^4}{4(4!)} + \frac{(2\pi)^6}{6(6!)} - \frac{(2\pi)^8}{8(8!)} + \cdots \right] \\ &\simeq 36.56\, I_o^2 \end{aligned} \tag{13.26}$$

The radiation resistance R_{rad} for the half-wave dipole antenna is readily obtained from eqs. (13.12) and (13.26) as

$$\boxed{R_{\text{rad}} = \frac{2P_{\text{rad}}}{I_o^2} = 73\ \Omega} \tag{13.27}$$

Note the significant increase in the radiation resistance of the half-wave dipole over that of the Hertzian dipole. Thus the half-wave dipole is capable of delivering greater amounts of power to space than the Hertzian dipole.

The total input impedance Z_{in} of the antenna is the impedance seen at the terminals of the antenna and is given by

$$Z_{in} = R_{in} + jX_{in} \tag{13.28}$$

where $R_{in} = R_{rad}$ for lossless antenna. Deriving the value of the reactance X_{in} involves a complicated procedure beyond the scope of this text. It is found that $X_{in} = 42.5\ \Omega$, so $Z_{in} = 73 + j42.5\ \Omega$ for a dipole length $\ell = \lambda/2$. The inductive reactance drops rapidly to zero as the length of the dipole is slightly reduced. For $\ell = 0.485\ \lambda$, the dipole is resonant, with $X_{in} = 0$. Thus in practice, a $\lambda/2$ dipole is designed such that X_{in} approaches zero and $Z_{in} \simeq 73\ \Omega$. This value of the radiation resistance of the $\lambda/2$ dipole antenna is the reason for the standard 75-Ω coaxial cable. Also, the value is easy to match to transmission lines. These factors in addition to the resonance property are the reasons for the dipole antenna's popularity and its extensive use.

13.4 QUARTER-WAVE MONOPOLE ANTENNA

Basically, the quarter-wave monopole antenna consists of one-half of a half-wave dipole antenna located on a conducting ground plane as in Figure 13.5. The monopole antenna is perpendicular to the plane, which is usually assumed to be infinite and perfectly conducting. It is fed by a coaxial cable connected to its base.

Using image theory of Section 6.6, we replace the infinite, perfectly conducting ground plane with the image of the monopole. The field produced in the region above the ground plane due to the $\lambda/4$ monopole with its image is the same as the field due to a $\lambda/2$ wave dipole. Thus eq. (13.19) holds for the $\lambda/4$ monopole. However, the integration in eq. (13.21) is only over the hemispherical surface above the ground plane (i.e., $0 \le \theta \le \pi/2$) because the monopole radiates only through that surface. Hence, the monopole radiates only half as much power as the dipole with the same current. Thus for a $\lambda/4$ monopole,

$$P_{rad} \simeq 18.28\, I_o^2 \tag{13.29}$$

and

$$R_{rad} = \frac{2P_{rad}}{I_o^2}$$

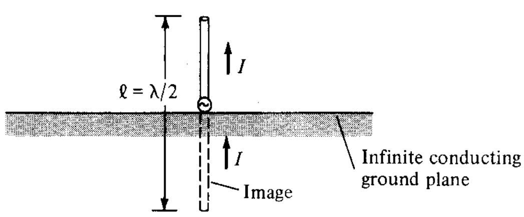

Figure 13.5 The monopole antenna.

or

$$\boxed{R_{rad} = 36.5\ \Omega} \tag{13.30}$$

By the same token, the total input impedance for a $\lambda/4$ monopole is $Z_{in} = 36.5 + j21.25\ \Omega$.

13.5 SMALL LOOP ANTENNA

The loop antenna is of practical importance. It is used as a directional finder (or search loop) in radiation detection and as a TV antenna for ultrahigh frequencies. The term "small" implies that the dimensions (such as ρ_o) of the loop are much smaller than λ.

Consider a small filamentary circular loop of radius ρ_o carrying a uniform current, $I_o \cos \omega t$, as in Figure 13.6. The loop may be regarded as an elemental magnetic dipole. The magnetic vector potential at the field point P due to the loop is

$$\mathbf{A} = \oint_L \frac{\mu[I]\, d\mathbf{l}}{4\pi r'} \tag{13.31}$$

where $[I] = I_o \cos(\omega t - \beta r') = \text{Re}\,[I_o e^{j(\omega t - \beta r')}]$. Substituting $[I]$ into eq. (13.31), we obtain $\mathbf{A}$ in phasor form as

$$\mathbf{A}_s = \frac{\mu I_o}{4\pi} \oint_L \frac{e^{-j\beta r'}}{r'}\, d\mathbf{l} \tag{13.32}$$

Evaluating this integral requires a lengthy procedure. It can be shown that for a small loop ($\rho_o \ll \lambda$), r' can be replaced by r in the denominator of eq. (13.32) and $\mathbf{A}_s$ has only ϕ-component given by

$$A_{\phi s} = \frac{\mu I_o S}{4\pi r^2}(1 + j\beta r)e^{-j\beta r} \sin\theta \tag{13.33}$$

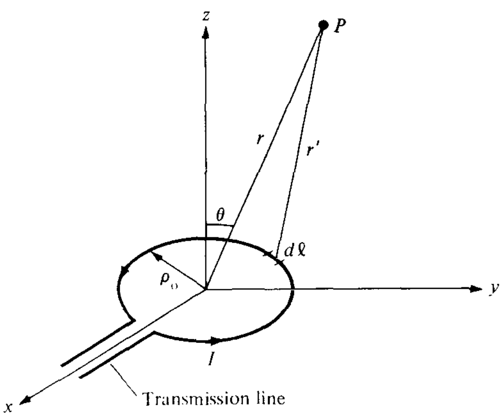

Figure 13.6 The small loop antenna.

where $S = \pi\rho_o^2$ = loop area. For a loop with N turns, $S = N\pi\rho_o^2$. Using the fact that $\mathbf{B}_s = \mu\mathbf{H}_s = \nabla \times \mathbf{A}_s$ and $\nabla \times \mathbf{H}_s = j\omega\varepsilon\mathbf{E}_s$, we obtain the electric and magnetic fields from eq. (13.33) as

$$E_{\phi s} = \frac{-j\omega\mu I_o S}{4\pi} \sin\theta \left[\frac{j\beta}{r} + \frac{1}{r^2}\right] e^{-j\beta r} \tag{13.34a}$$

$$H_{rs} = \frac{j\omega\mu I_o S}{2\pi\eta} \cos\theta \left[\frac{1}{r^2} - \frac{j}{\beta r^3}\right] e^{-j\beta r} \tag{13.34b}$$

$$H_{\theta s} = \frac{j\omega\mu I_o S}{4\pi\eta} \sin\theta \left[\frac{j\beta}{r} + \frac{1}{r^2} - \frac{j}{\beta r^3}\right] e^{-j\beta r} \tag{13.34c}$$

$$E_{rs} = E_{\theta s} = H_{\phi s} = 0 \tag{13.34d}$$

Comparing eqs. (13.5) and (13.6) with eq. (13.34), we observe the dual nature of the field due to an electric dipole of Figure 13.3 and the magnetic dipole of Figure 13.6 (see Table 8.2 also). At far field, only the $1/r$ term (the radiation term) in eq. (13.34) remains. Thus at far field,

$$\begin{aligned} E_{\phi s} &= \frac{\omega\mu I_o S}{4\pi r} \beta \sin\theta\, e^{-j\beta r} \\ &= \frac{\eta\pi I_o S}{r\lambda^2} \sin\theta\, e^{-j\beta r} \end{aligned}$$

or

$$\boxed{E_{\phi s} = \frac{120\pi^2 I_o}{r} \frac{S}{\lambda^2} \sin\theta\, e^{-j\beta r}, \qquad H_{\theta s} = -\frac{E_{\phi s}}{\eta}} \tag{13.35a}$$

$$E_{rs} = E_{\theta s} = H_{rs} = H_{\phi s} = 0 \tag{13.35b}$$

where $\eta = 120\pi$ for free space has been assumed. Though the far field expressions in eq. (13.35) are obtained for a small circular loop, they can be used for a small square loop with one turn ($S = a^2$), with N turns ($S = Na^2$) or any small loop provided that the loop dimensions are small ($d \leq \lambda/10$, where d is the largest dimension of the loop). It is left as an exercise to show that using eqs. (13.13a) and (13.35) gives the radiation resistance of a small loop antenna as

$$\boxed{R_{\text{rad}} = \frac{320\,\pi^4 S^2}{\lambda^4}} \tag{13.36}$$

EXAMPLE 13.1

A magnetic field strength of 5 μA/m is required at a point on $\theta = \pi/2$, 2 km from an antenna in air. Neglecting ohmic loss, how much power must the antenna transmit if it is

(a) A Hertzian dipole of length $\lambda/25$?

(b) A half-wave dipole?

(c) A quarter-wave monopole?

(d) A 10-turn loop antenna of radius $\rho_o = \lambda/20$?

Solution:

(a) For a Hertzian dipole,

$$|H_{\phi s}| = \frac{I_o \beta\, dl \sin\theta}{4\pi r}$$

where $dl = \lambda/25$ or $\beta\, dl = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{25} = \frac{2\pi}{25}$. Hence,

$$5 \times 10^{-6} = \frac{I_o \cdot \frac{2\pi}{25}(1)}{4\pi\,(2 \times 10^3)} = \frac{I_o}{10^5}$$

or

$$I_o = 0.5 \text{ A}$$

$$P_{rad} = 40\pi^2 \left[\frac{dl}{\lambda}\right]^2 I_o^2 = \frac{40\pi^2(0.5)^2}{(25)^2}$$
$$= 158 \text{ mW}$$

(b) For a $\lambda/2$ dipole,

$$|H_{\phi s}| = \frac{I_o \cos\left(\frac{\pi}{2}\cos\theta\right)}{2\pi r \sin\theta}$$

$$5 \times 10^{-6} = \frac{I_o \cdot 1}{2\pi\,(2 \times 10^3) \cdot (1)}$$

or

$$I_o = 20\pi \text{ mA}$$

$$P_{rad} = 1/2 I_o^2 R_{rad} = 1/2(20\pi)^2 \times 10^{-6}(73)$$
$$= 144 \text{ mW}$$

(c) For a λ/4 monopole,

$$I_o = 20\pi \text{ mA}$$

as in part (b).

$$P_{rad} = 1/2 I_o^2 R_{rad} = 1/2(20\pi)^2 \times 10^{-6}(36.56)$$
$$= 72 \text{ mW}$$

(d) For a loop antenna,

$$|H_{\theta s}| = \frac{\pi I_o}{r} \frac{S}{\lambda^2} \sin\theta$$

For a single turn, $S = \pi\rho_o^2$. For N-turn, $S = N\pi\rho_o^2$. Hence,

$$5 \times 10^{-6} = \frac{\pi I_o 10\pi}{2 \times 10^3} \left[\frac{\rho_o}{\lambda}\right]^2$$

or

$$I_o = \frac{10}{10\pi^2} \left[\frac{\lambda}{\rho_o}\right]^2 \times 10^{-3} = \frac{20^2}{\pi^2} \times 10^{-3}$$
$$= 40.53 \text{ mA}$$

$$R_{rad} = \frac{320\,\pi^4 S^2}{\lambda^4} = 320\,\pi^6 N^2 \left[\frac{\rho_o}{\lambda}\right]^4$$
$$= 320\,\pi^6 \times 100 \left[\frac{1}{20}\right]^4 = 192.3\ \Omega$$

$$P_{rad} = \frac{1}{2} I_o^2 R_{rad} = \frac{1}{2}(40.53)^2 \times 10^{-6}(192.3)$$
$$= 158 \text{ mW}$$

PRACTICE EXERCISE 13.1

A Hertzian dipole of length λ/100 is located at the origin and fed with a current of 0.25 sin $10^8 t$ A. Determine the magnetic field at

(a) $r = \lambda/5$, $\theta = 30°$

(b) $r = 200\lambda$, $\theta = 60°$

Answer: (a) 0.2119 sin $(10^8 t - 20.5°)$ $\mathbf{a}_\phi$ mA/m, (b) 0.2871 sin $(10^8 t + 90°)$ $\mathbf{a}_\phi$ μA/m.

EXAMPLE 13.2

An electric field strength of 10 μV/m is to be measured at an observation point $\theta = \pi/2$, 500 km from a half-wave (resonant) dipole antenna operating in air at 50 MHz.

(a) What is the length of the dipole?

(b) Calculate the current that must be fed to the antenna.

(c) Find the average power radiated by the antenna.

(d) If a transmission line with $Z_o = 75\ \Omega$ is connected to the antenna, determine the standing wave ratio.

Solution:

(a) The wavelength $\lambda = \dfrac{c}{f} = \dfrac{3 \times 10^8}{50 \times 10^6} = 6$ m.

Hence, the length of the half-dipole is $\ell = \dfrac{\lambda}{2} = 3$ m.

(b) From eq. (13.19),

$$|E_{\phi s}| = \frac{\eta_o I_o \cos\left(\frac{\pi}{2}\cos\theta\right)}{2\pi r \sin\theta}$$

or

$$\begin{aligned} I_o &= \frac{|E_{\theta s}|\, 2\pi r \sin\theta}{\eta_o \cos\left(\frac{\pi}{2}\cos\theta\right)} \\ &= \frac{10 \times 10^{-6}\, 2\pi\,(500 \times 10^3)\cdot(1)}{120\pi\,(1)} \\ &= 83.33 \text{ mA} \end{aligned}$$

(c)

$$R_{rad} = 73\ \Omega$$

$$\begin{aligned} P_{rad} &= \frac{1}{2} I_o^2 R_{rad} = \frac{1}{2}(83.33)^2 \times 10^{-6} \times 73 \\ &= 253.5 \text{ mW} \end{aligned}$$

(d)

$$\begin{aligned} \Gamma &= \frac{Z_L - Z_o}{Z_L + Z_o} \quad (Z_L = Z_{in} \text{ in this case}) \\ &= \frac{73 + j42.5 - 75}{73 + j42.5 + 75} = \frac{-2 + j42.5}{148 + j42.5} \\ &= \frac{42.55\angle 92.69^\circ}{153.98\angle 16.02^\circ} = 0.2763\angle 76.67^\circ \end{aligned}$$

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.2763}{1 - 0.2763} = 1.763$$

PRACTICE EXERCISE 13.2

Repeat Example 13.2 if the dipole antenna is replaced by a $\lambda/4$ monopole.

Answer: (a) 1.5m, (b) 83.33 mA, (c) 126.8 mW, (d) 2.265.

13.6 ANTENNA CHARACTERISTICS

Having considered the basic elementary antenna types, we now discuss some important characteristics of an antenna as a radiator of electromagnetic energy. These characteristics include: (a) antenna pattern, (b) radiation intensity, (c) directive gain, (d) power gain.

A. Antenna Patterns

An **antenna pattern** (or **radiation pattern**) is a three-dimensional plot of its radiation at far field.

When the amplitude of a specified component of the **E** field is plotted, it is called the *field pattern* or *voltage pattern.* When the square of the amplitude of **E** is plotted, it is called the *power pattern.* A three-dimensional plot of an antenna pattern is avoided by plotting separately the normalized $|E_s|$ versus θ for a constant ϕ (this is called an *E-plane pattern* or *vertical pattern*) and the normalized $|E_s|$ versus ϕ for $\theta = \pi/2$ (called the *H-plane pattern* or *horizontal pattern*). The normalization of $|E_s|$ is with respect to the maximum value of the $|E_s|$ so that the maximum value of the normalized $|E_s|$ is unity.

For the Hertzian dipole, for example, the normalized $|E_s|$ is obtained from eq. (13.7) as

$$f(\theta) = |\sin\theta| \tag{13.37}$$

which is independent of ϕ. From eq. (13.37), we obtain the E-plane pattern as the polar plot of $f(\theta)$ with θ varying from 0° to 180°. The result is shown in Figure 13.7(a). Note that the plot is symmetric about the z-axis ($\theta = 0$). For the H-plane pattern, we set $\theta = \pi/2$ so that $f(\theta) = 1$, which is circle of radius 1 as shown in Figure 13.7(b). When the two plots of Figures 13.7(a) and (b) are combined, we have a three-dimensional field pattern of Figure 13.7(c), which has the shape of a doughnut.

A plot of the time-average power, $|\mathcal{P}_{ave}| = \mathcal{P}_{ave}$, for a fixed distance r is the power pattern of the antenna. It is obtained by plotting separately $\mathcal{P}_{ave}$ versus θ for constant ϕ and $\mathcal{P}_{ave}$ versus ϕ for constant θ.

For the Hertzian dipole, the normalized power pattern is easily obtained from eqs. (13.37) or (13.9) as

$$f^2(\theta) = \sin^2\theta \tag{13.38}$$

which is sketched in Figure 13.8. Notice that Figures 13.7(b) and 13.8(b) show circles because $f(\theta)$ is independent of ϕ and that the value of OP in Figure 13.8(a) is the relative

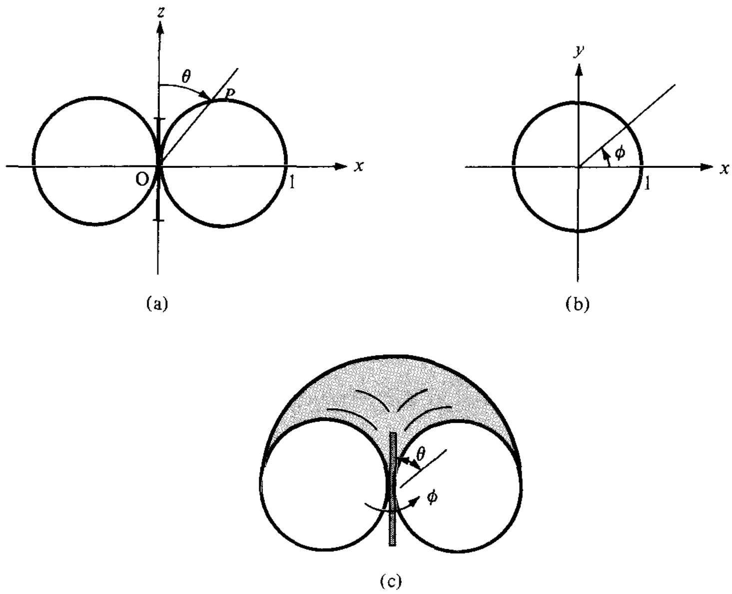

Figure 13.7 Field patterns of the Hertzian dipole: **(a)** normalized E-plane or vertical pattern (ϕ = constant = 0), **(b)** normalized H-plane or horizontal pattern ($\theta = \pi/2$), **(c)** three-dimensional pattern.

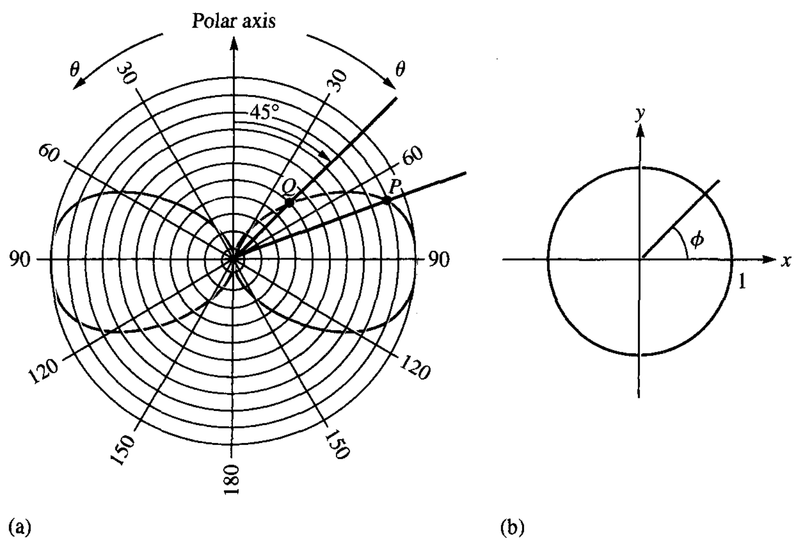

Figure 13.8 Power pattern of the Hertzian dipole: **(a)** ϕ = constant = 0; **(b)** θ = constant = $\pi/2$.

average power for that particular θ. Thus, at point Q ($\theta = 45°$), the average power is one-half the maximum average power (the maximum average power is at $\theta = \pi/2$).

B. Radiation Intensity

The radiation intensity of an antenna is defined as

$$\boxed{U(\theta, \phi) = r^2 \mathcal{P}_{ave}} \tag{13.39}$$

From eq. (13.39), the total average power radiated can be expressed as

$$\begin{aligned} P_{rad} &= \oint_S \mathcal{P}_{ave}\, dS = \oint_S \mathcal{P}_{ave}\, r^2 \sin\theta\, d\theta\, d\phi \\ &= \int_S U(\theta, \phi) \sin\theta\, d\theta\, d\phi \\ &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} U(\theta, \phi)\, d\Omega \end{aligned} \tag{13.40}$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$ is the *differential solid angle* in steradian (sr). Hence the radiation intensity $U(\theta, \phi)$ is measured in watts per steradian (W/sr). The average value of $U(\theta, \phi)$ is the total radiated power divided by 4π sr; that is,

$$\boxed{U_{ave} = \frac{P_{rad}}{4\pi}} \tag{13.41}$$

C. Directive Gain

Besides the antenna patterns described above, we are often interested in measurable quantities such as gain and directivity to determine the radiation characteristics of an antenna.

The **directive gain** $G_d(\theta, \phi)$ of an antenna is a measure of the concentration of the radiated power in a particular direction (θ, ϕ).

It may be regarded as the ability of the antenna to direct radiated power in a given direction. It is usually obtained as the ratio of radiation intensity in a given direction (θ, ϕ) to the average radiation intensity, that is

$$\boxed{G_d(\theta, \phi) = \frac{U(\theta, \phi)}{U_{ave}} = \frac{4\pi\, U(\theta, \phi)}{P_{rad}}} \tag{13.42}$$

By substituting eq. (13.39) into eq. (13.42), $\mathcal{P}_{\text{ave}}$ may be expressed in terms of directive gain as

$$\mathcal{P}_{\text{ave}} = \frac{G_d}{4\pi r^2} P_{\text{rad}} \tag{13.43}$$

The directive gain $G_d(\theta, \phi)$ depends on antenna pattern. For the Hertzian dipole (as well as for $\lambda/2$ dipole and $\lambda/4$ monopole), we notice from Figure 13.8 that $\mathcal{P}_{\text{ave}}$ is maximum at $\theta = \pi/2$ and minimum (zero) at $\theta = 0$ or π. Thus the Hertzian dipole radiates power in a direction broadside to its length. For an *isotropic* antenna (one that radiates equally in all directions), $G_d = 1$. However, such an antenna is not a physicality but an ideality.

The **directivity** D of an antenna is the ratio of the maximum radiation intensity to the average radiation intensity.

Obviously, D is the maximum directive gain G_d, max. Thus

$$\boxed{D = \frac{U_{\text{max}}}{U_{\text{ave}}} = G_d, \text{max}} \tag{13.44a}$$

or

$$D = \frac{4\pi\, U_{\text{max}}}{P_{\text{rad}}} \tag{13.44b}$$

$D = 1$ for an isotropic antenna; this is the smallest value D can have. For the Hertzian dipole,

$$G_d(\theta, \phi) = 1.5 \sin^2 \theta, \qquad D = 1.5. \tag{13.45}$$

For the $\lambda/2$ dipole,

$$G_d(\theta, \phi) = \frac{\eta}{\pi R_{\text{rad}}} f^2(\theta), \qquad D = 1.64 \tag{13.46}$$

where $\eta = 120\pi$, $R_{\text{rad}} = 73\ \Omega$, and

$$f(\theta) = \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \tag{13.47}$$

D. Power Gain

Our definition of the directive gain in eq. (13.42) does not account for the ohmic power loss P_ℓ of the antenna. P_ℓ is due to the fact that the antenna is made of a conductor with finite conductivity. As illustrated in Figure 13.9, if P_{in} is the total input power to the antenna,

$$\begin{aligned} P_{\text{in}} &= P_\ell + P_{\text{rad}} \\ &= \frac{1}{2} |I_{\text{in}}|^2 (R_\ell + R_{\text{rad}}) \end{aligned} \tag{13.48}$$

where I_{in} is the current at the input terminals and R_ℓ is the *loss* or *ohmic resistance* of the antenna. In other words, P_{in} is the power accepted by the antenna at its terminals during the radiation process, and P_{rad} is the power radiated by the antenna; the difference between the two powers is P_ℓ, the power dissipated within the antenna.

We define the *power gain* $G_p(\theta, \phi)$ of the antenna as

$$\boxed{G_p(\theta, \phi) = \frac{4\pi\, U(\theta, \phi)}{P_{\text{in}}}} \tag{13.49}$$

The ratio of the power gain in any specified direction to the directive gain in that direction is referred to as the *radiation efficiency* η_r of the antennas, that is

$$\eta_r = \frac{G_P}{G_d} = \frac{P_{\text{rad}}}{P_{\text{in}}}$$

Introducing eq. (13.48) leads to

$$\boxed{\eta_r = \frac{P_{\text{rad}}}{P_{\text{in}}} = \frac{R_{\text{rad}}}{R_{\text{rad}} + R_\ell}} \tag{13.50}$$

For many antennas, η_r is close to 100% so that $G_P \simeq G_d$. It is customary to express directivity and gain in decibels (dB). Thus

$$D\,(\text{dB}) = 10 \log_{10} D \tag{13.51a}$$

$$G\,(\text{dB}) = 10 \log_{10} G \tag{13.51b}$$

It should be mentioned at this point that the radiation patterns of an antenna are usually measured in the far field region. The far field region of an antenna is commonly taken to exist at distance $r \geq r_{\text{min}}$ where

$$r_{\text{min}} = \frac{2d^2}{\lambda} \tag{13.52}$$

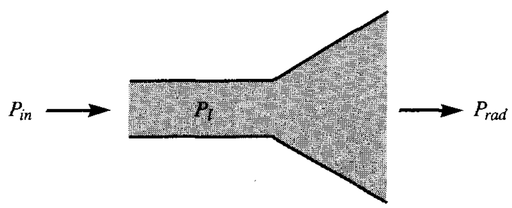

Figure 13.9 Relating P_{in}, P_ℓ, and P_{rad}.

and d is the largest dimension of the antenna. For example, $d = \ell$ for the electric dipole antenna and $d = 2\rho_o$ for the small loop antenna.

EXAMPLE 13.3

Show that the directive gain of the Hertzian dipole is

$$G_d(\theta, \phi) = 1.5 \sin^2 \theta$$

and that of the half-wave dipole is

$$G_d(\theta, \phi) = 1.64 \frac{\cos^2\left(\frac{\pi}{2} \cos \theta\right)}{\sin^2 \theta}$$

Solution:

From eq. (13.42),

$$G_d(\theta, \phi) = \frac{4\pi f^2(\theta)}{\int f^2(\theta)\, d\Omega}$$

(a) For the Hertzian dipole,

$$G_d(\theta, \phi) = \frac{4\pi \sin^2 \theta}{\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin^3 \theta \, d\theta \, d\phi} = \frac{4\pi \sin^2 \theta}{2\pi \, (4/3)}$$
$$= 1.5 \sin^2 \theta$$

as required.

(b) For the half-wave dipole,

$$G_d(\theta, \phi) = \frac{\dfrac{4\pi \cos^2\left(\frac{\pi}{2} \cos \theta\right)}{\sin^2 \theta}}{\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \dfrac{\cos^2\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta} \, d\theta \, d\phi}$$

From eq. (13.26), the integral in the denominator gives $2\pi(1.2188)$. Hence,

$$G_d(\theta, \phi) = \frac{4\pi \cos^2\left(\frac{\pi}{2} \cos \theta\right)}{\sin^2 \theta} \cdot \frac{1}{2\pi \, (1.2188)}$$
$$= 1.64 \frac{\cos^2\left(\frac{\pi}{2} \cos \theta\right)}{\sin^2 \theta}$$

as required.

PRACTICE EXERCISE 13.3

Calculate the directivity of

(a) The Hertzian monopole

(b) The quarter-wave monopole

Answer: (a) 3, (b) 3.28.

EXAMPLE 13.4

Determine the electric field intensity at a distance of 10 km from an antenna having a directive gain of 5 dB and radiating a total power of 20 kW.

Solution:

$$5 = G_d\,(\text{dB}) = 10 \log_{10} G_d$$

or

$$0.5 = \log_{10} G_d \rightarrow G_d = 10^{0.5} = 3.162$$

From eq. (13.43),

$$\mathcal{P}_{\text{ave}} = \frac{G_d P_{\text{rad}}}{4\pi r^2}$$

But

$$\mathcal{P}_{\text{ave}} = \frac{|E_s|^2}{2\eta}$$

Hence,

$$|E_s|^2 = \frac{\eta G_d P_{\text{rad}}}{2\pi r^2} = \frac{120\pi(3.162)(20 \times 10^3)}{2\pi\,[10 \times 10^3]^2}$$

$$|E_s| = 0.1948 \text{ V/m}$$

PRACTICE EXERCISE 13.4

A certain antenna with an efficiency of 95% has maximum radiation intensity of 0.5 W/sr. Calculate its directivity when

(a) The input power is 0.4 W

(b) The radiated power is 0.3 W

Answer: (a) 16.53, (b) 20.94.

EXAMPLE 13.5

The radiation intensity of a certain antenna is

$$U(\theta, \phi) = \begin{cases} 2 \sin \theta \sin^3 \phi, & 0 \le \theta \le \pi, 0 \le \phi \le \pi \\ 0, & \text{elsewhere} \end{cases}$$

Determine the directivity of the antenna.

Solution:

The directivity is defined as

$$D = \frac{U_{\text{max}}}{U_{\text{ave}}}$$

From the given U,

$$U_{\text{max}} = 2$$

$$\begin{aligned} U_{\text{ave}} &= \frac{1}{4\pi} \int U \, d\Omega \; (=P_{\text{rad}}/4\pi) \\ &= \frac{1}{4\pi} \int_{\phi=0}^{\pi} \int_{\theta=0}^{\pi} 2 \sin \theta \sin^3 \phi \sin \theta \, d\theta \, d\phi \\ &= \frac{1}{2\pi} \int_0^{\pi} \sin^2 \theta \, d\theta \int_0^{\pi} \sin^3 \phi \, d\phi \\ &= \frac{1}{2\pi} \int_0^{\pi} \frac{1}{2}(1 - \cos 2\theta) \, d\theta \int_0^{\pi} (1 - \cos^2 \phi) \, d(-\cos \phi) \\ &= \frac{1}{2\pi} \frac{1}{2} \left(\theta - \frac{\sin 2\theta}{2} \right) \Bigg|_0^{\pi} \left(\frac{\cos^3 \phi}{3} - \cos \phi \right) \Bigg|_0^{\pi} \\ &= \frac{1}{2\pi} \left(\frac{\pi}{2} \right) \left(\frac{4}{3} \right) = \frac{1}{3} \end{aligned}$$

Hence

$$D = \frac{2}{(1/3)} = 6$$

PRACTICE EXERCISE 13.5

Evaluate the directivity of an antenna with normalized radiation intensity

$$U(\theta, \phi) = \begin{cases} \sin \theta, & 0 \le \theta \le \pi/2, 0 \le \phi \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

Answer: 2.546.

13.7 ANTENNA ARRAYS

In many practical applications (e.g., in an AM broadcast station), it is necessary to design antennas with more energy radiated in some particular directions and less in other directions. This is tantamount to requiring that the radiation pattern be concentrated in the direction of interest. This is hardly achievable with a single antenna element. An antenna array is used to obtain greater directivity than can be obtained with a single antenna element.

> An **antenna array** is a group of radiating elements arranged so as to produce some particular radiation characteristics.

It is practical and convenient that the array consists of identical elements but this is not fundamentally required. We shall consider the simplest case of a two-element array and extend our results to the more complicated, general case of an N-element array.

Consider an antenna consisting of two Hertzian dipoles placed in free space along the z-axis but oriented parallel to the x-axis as depicted in Figure 13.10. We assume that the dipole at $(0, 0, d/2)$ carries current $I_{1s} = I_o\underline{/\alpha}$ and the one at $(0, 0, -d/2)$ carries current $I_{2s} = I_o\underline{/0}$, where α is the phase difference between the two currents. By varying the spacing d and phase difference α, the fields from the array can be made to interfere constructively (add) in certain directions of interest and interfere destructively (cancel) in other directions. The total electric field at point P is the vector sum of the fields due to the individual elements. If P is in the far field zone, we obtain the total electric field at P from eq. (13.7a) as

$$\begin{aligned}\mathbf{E}_s &= \mathbf{E}_{1s} + \mathbf{E}_{2s} \\ &= \frac{j\eta\beta I_o dl}{4\pi}\left[\cos\theta_1 \frac{e^{-j\beta r_1}}{r_1} e^{j\alpha}\,\mathbf{a}_{\theta_1} + \cos\theta_2 \frac{e^{-j\beta r_2}}{r_2}\,\mathbf{a}_{\theta_2}\right]\end{aligned} \tag{13.53}$$

Note that $\sin\theta$ in eq. (13.7a) has been replaced by $\cos\theta$ since the element of Figure 13.3 is z-directed whereas those in Figure 13.10 are x-directed. Since P is far from the array,

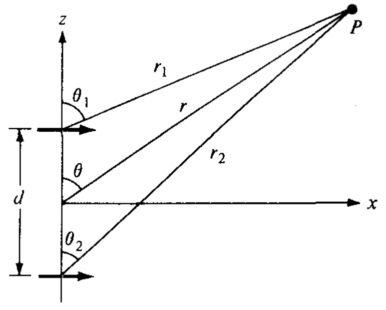

Figure 13.10 A two-element array.

$\theta_1 \simeq \theta \simeq \theta_2$ and $\mathbf{a}_{\theta_1} \simeq \mathbf{a}_\theta \simeq \mathbf{a}_{\theta_2}$. In the amplitude, we can set $r_1 \simeq r \approx r_2$ but in the phase, we use

$$r_1 \simeq r - \frac{d}{2}\cos\theta \tag{13.54a}$$

$$r_2 \simeq r + \frac{d}{2}\cos\theta \tag{13.54b}$$

Thus eq. (13.53) becomes

$$\begin{aligned}\mathbf{E}_s &= \frac{j\eta\beta I_o\, dl}{4\pi r}\cos\theta\, e^{-j\beta r}e^{j\alpha/2}[e^{j(\beta d\cos\theta)/2}e^{j\alpha/2} + e^{-j(\beta d\cos\theta)/2}e^{-j\alpha/2}]\mathbf{a}_\theta \\ &= \frac{j\eta\beta I_o\, dl}{4\pi r}\cos\theta\, e^{-j\beta r}e^{j\alpha/2}2\cos\left[\frac{1}{2}(\beta d\cos\theta + \alpha)\right]\mathbf{a}_\theta\end{aligned} \tag{13.55}$$

Comparing this with eq. (13.7a) shows that the total field of an array is equal to the field of single element located at the origin multiplied by an *array factor* given by

$$AF = 2\cos\left[\frac{1}{2}(\beta d\cos\theta + \alpha)\right]e^{j\alpha/2} \tag{13.56}$$

Thus, in general, the far field due to a two-element array is given by

$$\mathbf{E}\text{ (total)} = (\mathbf{E}\text{ due to single element at origin}) \times (\text{array factor}) \tag{13.57}$$

Also, from eq. (13.55), note that $|\cos\theta|$ is the radiation pattern due to a single element whereas the normalized array factor, $|\cos[1/2(\beta d\cos\theta + \alpha)]|$, is the radiation pattern of the array if the elements were isotropic. These may be regarded as "unit pattern" and "group pattern," respectively. Thus the "resultant pattern" is the product of the unit pattern and the group pattern, that is,

$$\text{Resultant pattern} = \text{Unit pattern} \times \text{Group pattern} \tag{13.58}$$

This is known as *pattern multiplication.* It is possible to sketch, almost by inspection, the pattern of an array by pattern multiplication. It is, therefore, a useful tool in the design of an array. We should note that while the unit pattern depends on the type of elements the array is comprised of, the group pattern is independent of the element type so long as the spacing d and phase difference α, and the orientation of the elements remain the same.

Let us now extend the results on the two-element array to the general case of an N-element array shown in Figure 13.11. We assume that the array is *linear* in that the elements are spaced equally along a straight line and lie along the z-axis. Also, we assume that the array is *uniform* so that each element is fed with current of the same magnitude but of progressive phase shift α, that is, $I_{1s} = I_o\angle 0$, $I_{2s} = I_o\angle\alpha$, $I_{3s} = I_o\angle 2\alpha$, and so on. We are mainly interested in finding the array factor; the far field can easily be found from eq.

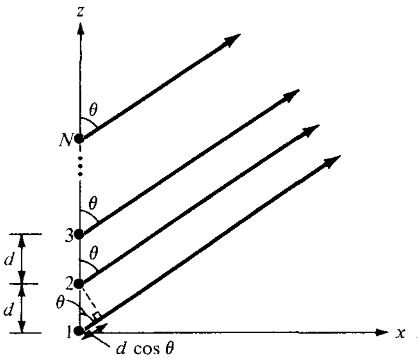

Figure 13.11 An N-element uniform linear array.

(13.57) once the array factor is known. For the uniform linear array, the array factor is the sum of the contributions by all the elements. Thus,

$$AF = 1 + e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \cdots + e^{j(N-1)\psi} \tag{13.59}$$

where

$$\psi = \beta d \cos\theta + \alpha \tag{13.60}$$

In eq. (13.60), $\beta = 2\pi/\lambda$, d and α are, respectively, the spacing and interelement phase shift. Notice that the right-hand side of eq. (13.59) is a geometric series of the form

$$1 + x + x^2 + x^3 + \cdots + x^{N-1} = \frac{1 - x^N}{1 - x} \tag{13.61}$$

Hence eq. (13.59) becomes

$$AF = \frac{1 - e^{jN\psi}}{1 - e^{j\psi}} \tag{13.62}$$

which can be written as

$$\begin{aligned} AF &= \frac{e^{jN\psi} - 1}{e^{j\psi} - 1} = \frac{e^{jN\psi/2}}{e^{j\psi/2}} \frac{e^{jN\psi/2} - e^{-jN\psi/2}}{e^{j\psi/2} - e^{-j\psi/2}} \\ &= e^{j(N-1)\psi/2} \frac{\sin(N\psi/2)}{\sin(\psi/2)} \end{aligned} \tag{13.63}$$

The phase factor $e^{j(N-1)\psi/2}$ would not be present if the array were centered about the origin. Neglecting this unimportant term,

$$\boxed{AF = \frac{\sin \dfrac{N\psi}{2}}{\sin \dfrac{\psi}{2}}, \qquad \psi = \beta d \cos\theta + \alpha} \tag{13.64}$$

Note that this equation reduces to eq. (13.56) when $N = 2$ as expected. Also, note the following:

1. AF has the maximum value of N; thus the normalized AF is obtained by dividing AF by N. The principal maximum occurs when $\psi = 0$, that is

$$0 = \beta d \cos\theta + \alpha \qquad \text{or} \qquad \cos\theta = -\frac{\alpha}{\beta d} \tag{13.65}$$

2. AF has *nulls* (or *zeros*) when $AF = 0$, that is

$$\frac{N\psi}{2} = \pm k\pi, \qquad k = 1, 2, 3, \ldots \tag{13.66}$$

where k is not a multiple of N.

3. A *broadside* array has its maximum radiation directed normal to the axis of the array, that is, $\psi = 0$, $\theta = 90°$ so that $\alpha = 0$.
4. An *end-fire* array has its maximum radiation directed along the axis of the array, that is, $\psi = 0$, $\theta = \begin{bmatrix} 0 \\ \pi \end{bmatrix}$ so that $\alpha = \begin{bmatrix} -\beta d \\ \beta d \end{bmatrix}$

These points are helpful in plotting AF. For $N = 2$, 3, and 4, the plots of AF are sketched in Figure 13.12.

EXAMPLE 13.6

For the two-element antenna array of Figure 13.10, sketch the normalized field pattern when the currents are:

(a) Fed in phase ($\alpha = 0$), $d = \lambda/2$

(b) Fed 90° out of phase ($\alpha = \pi/2$), $d = \lambda/4$

Solution:

The normalized field of the array is obtained from eqs. (13.55) to (13.57) as

$$f(\theta) = \left|\cos\theta \cos\left[\frac{1}{2}(\beta d \cos\theta + \alpha)\right]\right|$$

(a) If $\alpha = 0$, $d = \lambda/2$, $\beta d = \dfrac{2\pi}{\lambda}\dfrac{\lambda}{2} = \pi$. Hence,

$$\begin{array}{ccccc} f(\theta) & = & |\cos\theta| & & \left|\cos\dfrac{\pi}{2}(\cos\theta)\right| \\ \downarrow & & \downarrow & & \downarrow \\ \text{resultant} & = & \text{unit} & \times & \text{group} \\ \text{pattern} & & \text{pattern} & & \text{pattern} \end{array}$$

The sketch of the unit pattern is straightforward. It is merely a rotated version of that in Figure 13.7(a) for the Hertzian dipole and is shown in Figure 13.13(a). To sketch a

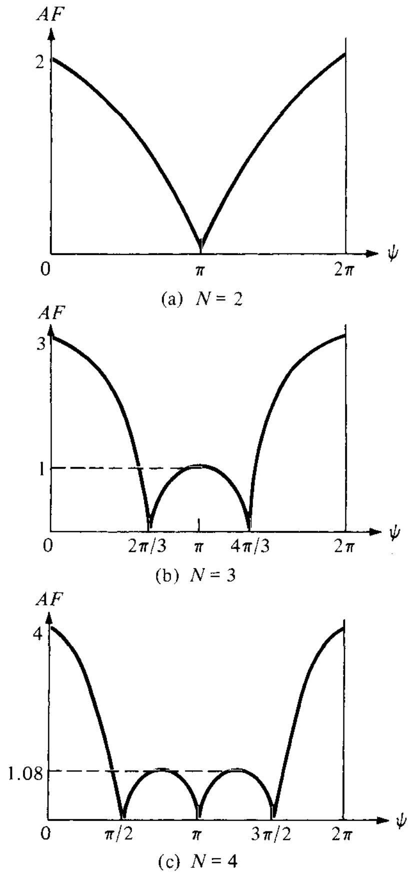

Figure 13.12 Array factor for uniform linear array.

group pattern requires that we first determine its nulls and maxima. For the nulls (or zeros),

$$\cos\left(\frac{\pi}{2}\cos\theta\right) = 0 \rightarrow \frac{\pi}{2}\cos\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$$

or

$$\theta = 0°, 180°$$

For the maxima,

$$\cos\left(\frac{\pi}{2}\cos\theta\right) = 1 \rightarrow \cos\theta = 0$$

or

$$\theta = 90°$$

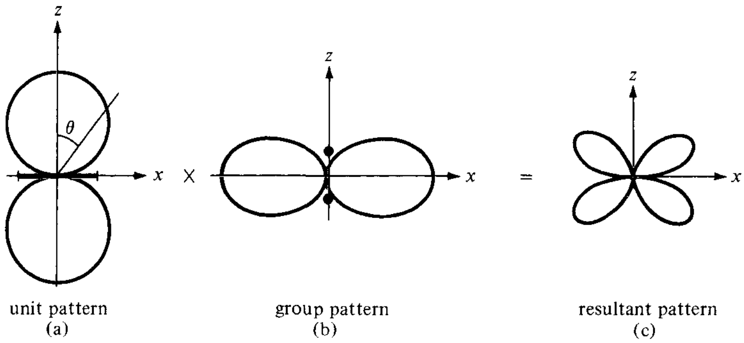

Figure 13.13 For Example 13.6(a); field patterns in the plane containing the axes of the elements.

The group pattern is as shown in Figure 13.12(b). It is the polar plot obtained by sketching $\left|\cos\left(\frac{\pi}{2}\cos\theta\right)\right|$ for $\theta = 0°, 5°, 10°, 15°, \ldots, 360°$ and incorporating the nulls and maxima at $\theta = 0°$, $180°$ and $\theta = 90°$, respectively. Multiplying Figure 13.13(a) with Figure 13.13(b) gives the resultant pattern in Figure 13.13(c). It should be observed that the field patterns in Figure 13.13 are in the plane containing the axes of the elements. Note that: (1) In the yz-plane, which is normal to the axes of the elements, the unit pattern (= 1) is a circle [see Figure 13.7(b)] while the group pattern remains as in Figure 13.13(b); therefore, the resultant pattern is the same as the group pattern in this case. (2) In the xy-plane, $\theta = \pi/2$, so the unit pattern vanishes while the group pattern (= 1) is a circle.

(b) If $\alpha = \pi/2$, $d = \lambda/4$, and $\beta d = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$

$$\underset{\substack{\text{resultant}\\ \text{pattern}}}{\underbrace{f(\theta)}} = \underset{\substack{\text{unit}\\ \text{pattern}}}{\underbrace{|\cos\theta|}} \times \underset{\substack{\text{group}\\ \text{pattern}}}{\underbrace{\left|\cos\frac{\pi}{4}(\cos\theta + 1)\right|}}$$

The unit pattern remains as in Figure 13.13(a). For the group pattern, the null occurs when

$$\cos\frac{\pi}{4}(1 + \cos\theta) = 0 \rightarrow \frac{\pi}{4}(1 + \cos\theta) = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$$

or

$$\cos\theta = 1 \rightarrow \theta = 0$$

The maxima and minima occur when

$$\frac{d}{d\theta}\left[\cos\frac{\pi}{4}(1 + \cos\theta)\right] = 0 \rightarrow \sin\theta\sin\frac{\pi}{4}(1 + \cos\theta) = 0$$

$$\sin\theta = 0 \rightarrow \theta = 0°, 180°$$

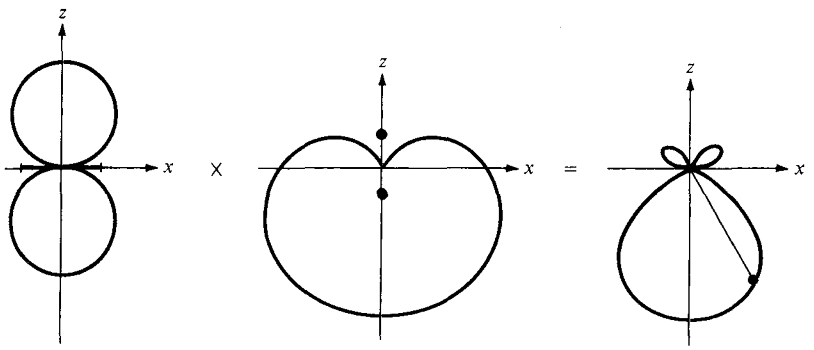

Figure 13.14 For Example 13.6(b); field patterns in the plane containing the axes of the elements.

and

$$\sin \frac{\pi}{4}(1 + \cos \theta) = 0 \rightarrow \cos \theta = -1 \quad \text{or} \quad \theta = 180°$$

Each field pattern is obtained by varying $\theta = 0°, 5°, 10°, 15°, \ldots, 180°$. Note that $\theta = 180°$ corresponds to the maximum value of AF, whereas $\theta = 0°$ corresponds to the null. Thus the unit, group, and resultant patterns in the plane containing the axes of the elements are shown in Figure 13.14. Observe from the group patterns that the broadside array ($\alpha = 0$) in Figure 13.13 is bidirectional while the end-fire array ($\alpha = \beta d$) in Figure 13.14 is unidirectional.

PRACTICE EXERCISE 13.6

Repeat Example 13.6 for cases when:

(a) $\alpha = \pi, d = \lambda/2$, (b) $\alpha = -\pi/2, d = \lambda/4$.

Answer: See Figure 13.15.

EXAMPLE 13.7

Consider a three-element array that has current ratios 1:2:1 as in Figure 13.16(a). Sketch the group pattern in the plane containing the axes of the elements.

Solution:

For the purpose of analysis, we split the middle element in Figure 13.16(a) carrying current $2I\angle 0°$ into two elements each carrying current $I\angle 0°$. This results in four elements instead of three as shown in Figure 13.16(b). If we consider elements 1 and 2 as a group and elements 3 and 4 as another group, we have a two-element array of Figure 13.16(c). Each

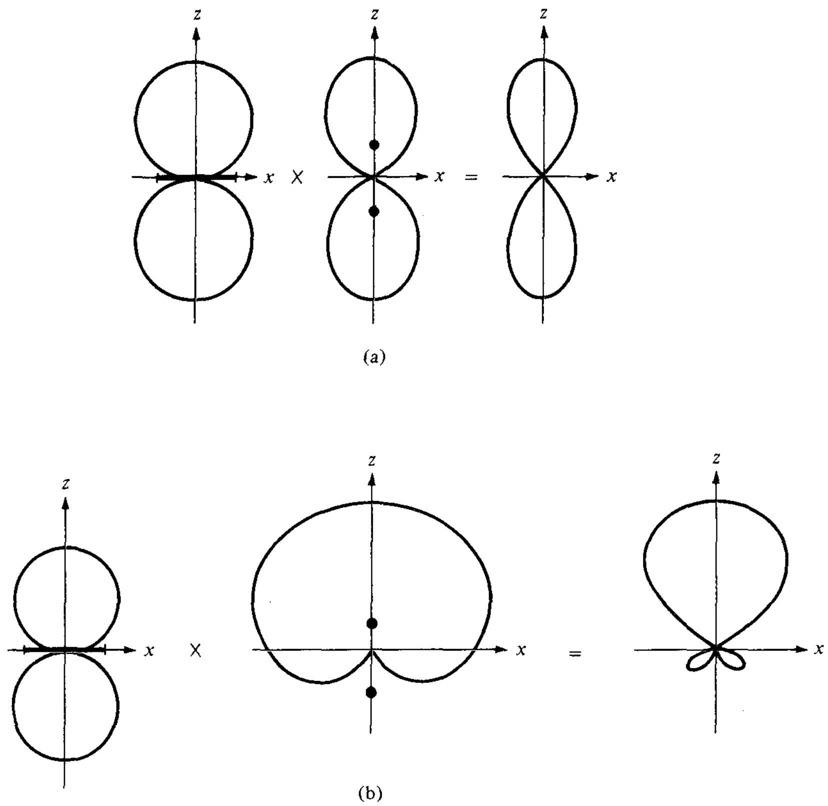

Figure 13.15 For Practice Exercise 13.6.

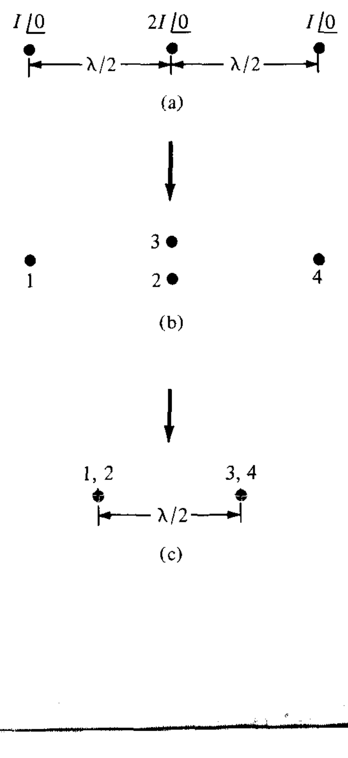

Figure 13.16 For Example 13.7: **(a)** a three-element array with current ratios 1 : 2 : 1; **(b)** and **(c)** equivalent two-element arrays.

group is a two-element array with $d = \lambda/2$, $\alpha = 0$, that the group pattern of the two-element array (or the unit pattern for the three-element array) is as shown in Figure 13.13(b). The two groups form a two-element array similar to Example 13.6(a) with $d = \lambda/2$, $\alpha = 0$, so the group pattern is the same as that in Figure 13.13(b). Thus, in this case, both the unit and group patterns are the same pattern in Figure 13.13(b). The resultant group pattern is obtained in Figure 13.17(c). We should note that the pattern in Figure 13.17(c) is not the resultant pattern but the group pattern of the three-element array. The resultant group pattern of the array is Figure 13.17(c) multiplied by the field pattern of the element type.

An alternative method of obtaining the resultant group pattern of the three-element array of Figure 13.16 is following similar steps taken to obtain eq. (13.59). We obtain the normalized array factor (or the group pattern) as

$$\begin{aligned}(AF)_n &= \frac{1}{4}|1 + 2e^{j\psi} + e^{j2\psi}| \\ &= \frac{1}{4}|e^{j\psi}||2 + e^{-j\psi} + e^{j\psi}| \\ &= \frac{1}{2}|1 + \cos\psi| = \left|\cos\frac{\psi}{2}\right|^2\end{aligned}$$

where $\psi = \beta d\cos\theta + \alpha$ if the elements are placed along the z-axis but oriented parallel to the x-axis. Since $\alpha = 0$, $d = \lambda/2$, $\beta d = \dfrac{2\pi}{\lambda}\cdot\dfrac{\lambda}{2} = \pi$,

$$(AF)_n = \left|\cos\left(\frac{\pi}{2}\cos\theta\right)\right|^2$$

$$\underset{\substack{\text{resultant}\\ \text{group pattern}}}{\underset{\downarrow}{(AF)_n}} = \underset{\substack{\text{unit}\\ \text{pattern}}}{\underset{\downarrow}{\left|\cos\left(\frac{\pi}{2}\cos\theta\right)\right|}} \times \underset{\substack{\text{group}\\ \text{pattern}}}{\underset{\downarrow}{\left|\cos\left(\frac{\pi}{2}\cos\theta\right)\right|}}$$

The sketch of these patterns is exactly what is in Figure 13.17.

If two three-element arrays in Figure 13.16(a) are displaced by $\lambda/2$, we obtain a four-element array with current ratios 1:3:3:1 as in Figure 13.18. Two of such four-element

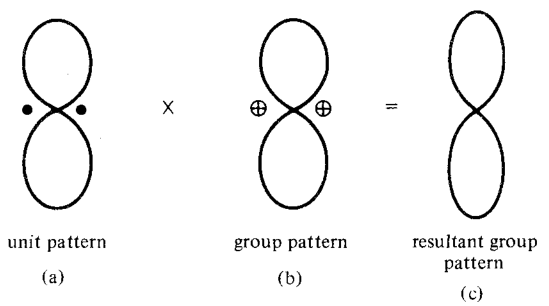

Figure 13.17 For Example 13.7; obtaining the resultant group pattern of the three-element array of Figure 13.16(a).

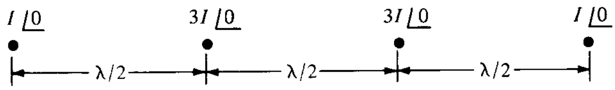

Figure 13.18 A four-element array with current ratios 1 : 3 : 3 : 1; for Practice Exercise 13.7.

arrays, displaced by $\lambda/2$, give a five-element array with current ratios 1 : 4 : 6 : 4 : 1. Continuing this process results in an N-element array, spaced $\lambda/2$ and $(N - 1)\lambda/2$ long, whose current ratios are the binomial coefficients. Such an array is called a linear *binomial array.*

PRACTICE EXERCISE 13.7

(a) Sketch the resultant group pattern for the four-element array with current ratios 1 : 3 : 3 : 1 shown in Figure 13.18.

(b) Derive an expression for the group pattern of a linear binomial array of N elements. Assume that the elements are placed along the z-axis, oriented parallel to the x-axis with spacing d and interelement phase shift α.

Answer: (a) See Figure 13.19, (b) $\left|\cos\frac{\psi}{2}\right|^{N-1}$, where $\psi = \beta d \cos\theta + \alpha$.

Figure 13.19 For Practice Exercise 13.7(a).

†13.8 EFFECTIVE AREA AND THE FRIIS EQUATION

In a situation where the incoming EM wave is normal to the entire surface of a receiving antenna, the power received is

$$P_r = \int \mathcal{P}_{\text{ave}} \cdot d\mathbf{S} = \mathcal{P}_{\text{ave}}\, S \tag{13.67}$$

But in most cases, the incoming EM wave is not normal to the entire surface of the antenna. This necessitates the idea of the effective area of a receiving antenna.

The concept of effective area or effective aperture (receiving cross section of an antenna) is usually employed in the analysis of receiving antennas.

The effective area A_e of a receiving antenna is the ratio of the time-average power received P_r (or delivered to the load, to be strict) to the time-average power density $\mathcal{P}_{\text{ave}}$ of the incident wave at the antenna.

That is

$$A_e = \frac{P_r}{\mathcal{P}_{ave}} \tag{13.68}$$

From eq. (13.68), we notice that the effective area is a measure of the ability of the antenna to extract energy from a passing EM wave.

Let us derive the formula for calculating the effective area of the Hertzian dipole acting as a receiving antenna. The Thevenin equivalent circuit for the receiving antenna is shown in Figure 13.20, where V_{oc} is the open-circuit voltage induced on the antenna terminals, $Z_{in} = R_{rad} + jX_{in}$ is the antenna impedance, and $Z_L = R_L + jX_L$ is the external load impedance, which might be the input impedance to the transmission line feeding the antenna. For maximum power transfer, $Z_L = Z_{in}^*$ and $X_L = -X_{in}$. The time-average power delivered to the matched load is therefore

$$\begin{aligned} P_r &= \frac{1}{2}\left[\frac{|V_{oc}|}{2R_{rad}}\right]^2 R_{rad} \\ &= \frac{|V_{oc}|^2}{8\,R_{rad}} \end{aligned} \tag{13.69}$$

For the Hertzian dipole, $R_{rad} = 80\pi^2(dl/\lambda)^2$ and $V_{oc} = E\,dl$ where E is the effective field strength parallel to the dipole axis. Hence, eq. (13.69) becomes

$$P_r = \frac{E^2\lambda^2}{640\pi^2} \tag{13.70}$$

The time-average power at the antenna is

$$\mathcal{P}_{ave} = \frac{E^2}{2\eta_o} = \frac{E^2}{240\pi} \tag{13.71}$$

Inserting eqs. (13.70) and (13.71) in eq. (13.68) gives

$$A_e = \frac{3\lambda^2}{8\pi} = 1.5\frac{\lambda^2}{4\pi}$$

or

$$A_e = \frac{\lambda^2}{4\pi}D \tag{13.72}$$

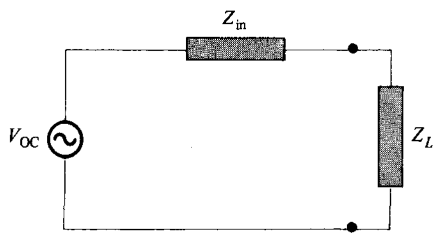

Figure 13.20 Thevenin equivalent of a receiving antenna.

where $D = 1.5$ is the directivity of the Hertzian dipole. Although eq. (13.72) was derived for the Hertzian dipole, it holds for any antenna if D is replaced by $G_d(\theta, \phi)$. Thus, in general

$$\boxed{A_e = \frac{\lambda^2}{4\pi} G_d(\theta, \phi)} \tag{13.73}$$

Now suppose we have two antennas separated by distance r in free space as shown in Figure 13.21. The transmitting antenna has effective area A_{et} and directive gain G_{dt}, and transmits a total power P_t $(= P_{\text{rad}})$. The receiving antenna has effective area of A_{er} and directive gain G_{dr}, and receives a total power of P_r. At the transmitter,

$$G_{dt} = \frac{4\pi U}{P_t} = \frac{4\pi r^2 \mathcal{P}_{\text{ave}}}{P_t}$$

or

$$\mathcal{P}_{\text{ave}} = \frac{P_t}{4\pi r^2} G_{dt} \tag{13.74}$$

By applying eqs. (13.68) and (13.73), we obtain the time-average power received as

$$P_r = \mathcal{P}_{\text{ave}} A_{er} = \frac{\lambda^2}{4\pi} G_{dr} \mathcal{P}_{\text{ave}} \tag{13.75}$$

Substituting eq. (13.74) into eq. (13.75) results in

$$\boxed{P_r = G_{dr} G_{dt} \left[\frac{\lambda}{4\pi r} \right]^2 P_t} \tag{13.76}$$

This is referred to as the *Friis transmission formula*. It relates the power received by one antenna to the power transmitted by the other, provided that the two antennas are separated by $r > 2d^2/\lambda$, where d is the largest dimension of either antenna [see eq. 13.52)]. Therefore, in order to apply the Friis equation, we must make sure that the two antennas are in the far field of each other.

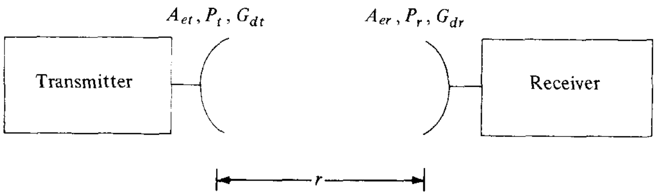

Figure 13.21 Transmitting and receiving antennas in free space.

EXAMPLE 13.8

Find the maximum effective area of a $\lambda/2$ wire dipole operating at 30 MHz. How much power is received with an incident plane wave of strength 2 mV/m.

Solution:

$$A_e = \frac{\lambda^2}{4\pi} G_d(\theta, \phi)$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{30 \times 10^6} = 10 \text{ m}$$

$$G_d(\theta, \phi) = \frac{\eta}{\pi R_{\text{rad}}} f^2(\theta) = \frac{120\pi}{73\pi} f^2(\theta) = 1.64 f^2(\theta)$$

$$G_d(\theta, \phi)_{\text{max}} = 1.64$$

$$A_{e,\text{max}} = \frac{10^2}{4\pi}(1.64) = 13.05 \text{ m}^2$$

$$P_r = \mathcal{P}_{\text{ave}} A_e = \frac{E_o^2}{2\eta} A_e$$

$$= \frac{(2 \times 10^{-3})^2}{240\pi} 13.05 = 71.62 \text{ nW}$$

PRACTICE EXERCISE 13.8

Determine the maximum effective area of a Hertzian dipole of length 10 cm operating at 10 MHz. If the antenna receives 3 μW of power, what is the power density of the incident wave?

Answer: 1.074 m^2, 2.793 μW/m^2

EXAMPLE 13.9

The transmitting and receiving antennas are separated by a distance of 200 λ and have directive gains of 25 and 18 dB, respectively. If 5 mW of power is to be received, calculate the minimum transmitted power.

Solution:

Given that G_{dt} (dB) $= 25$ dB $= 10 \log_{10} G_{dt}$,

$$G_{dt} = 10^{2.5} = 316.23$$

Similarly,

$$G_{dr}\,(dB) = 18 \text{ db} \qquad \text{or} \qquad G_{dr} = 10^{1.8} = 63.1$$

Using the Friis equation, we have

$$P_r = G_{dr}G_{dt}\left[\frac{\lambda}{4\pi r}\right]^2 P_t$$

or

$$\begin{aligned} P_t &= P_r\left[\frac{4\pi r}{\lambda}\right]^2 \frac{1}{G_{dr}G_{dt}} \\ &= 5 \times 10^{-3}\left[\frac{4\pi \times 200\,\lambda}{\lambda}\right]^2 \frac{1}{(63.1)(316.23)} \\ &= 1.583 \text{ W} \end{aligned}$$

PRACTICE EXERCISE 13.9

An antenna in air radiates a total power of 100 kW so that a maximum radiated electric field strength of 12 mV/m is measured 20 km from the antenna. Find: (a) its directivity in dB, (b) its maximum power gain if $\eta_r = 98\%$.

Answer: (a) 3.34 dB, (b) 2.117.

†13.9 THE RADAR EQUATION

Radars are electromagnetic devices used for detection and location of objects. The term *radar* is derived from the phrase *ra*dio *d*etection *a*nd *r*anging. In a typical radar system shown in Figure 13.22(a), pulses of EM energy are transmitted to a distant object. The same antenna is used for transmitting and receiving, so the time interval between the transmitted and reflected pulses is used to determine the distance of the target. If r is the dis-

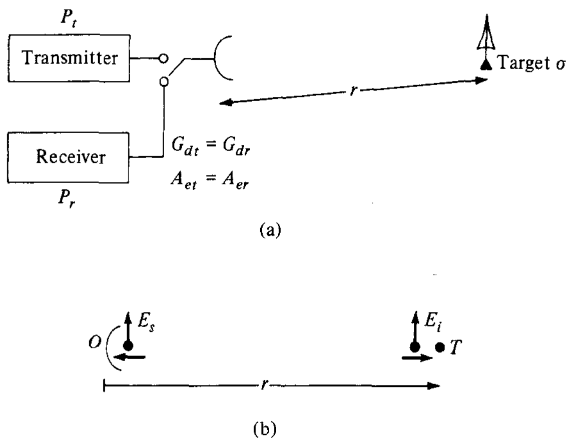

Figure 13.22 **(a)** Typical radar system, **(b)** simplification of the system in (a) for calculating the target cross section σ.

tance between the radar and target and c is the speed of light, the elapsed time between the transmitted and received pulse is $2r/c$. By measuring the elapsed time, r is determined.

The ability of the target to scatter (or reflect) energy is characterized by the *scattering cross section* σ (also called the *radar cross section*) of the target. The scattering cross section has the units of area and can be measured experimentally.

The **scattering cross section** is the equivalent area intercepting that amount of power that, when scattering isotropically, produces at the radar a power density, which is equal to that scattered (or reflected) by the actual target.

That is,

$$\mathcal{P}_s = \lim_{r \to \infty} \left[\frac{\sigma \mathcal{P}_i}{4\pi r^2} \right]$$

or

$$\sigma = \lim_{r \to \infty} 4\pi r^2 \frac{\mathcal{P}_s}{\mathcal{P}_i} \tag{13.77}$$

where $\mathcal{P}_i$ is the incident power density at the target T while $\mathcal{P}_s$ is the scattered power density at the transreceiver O as in Figure 13.22(b).

From eq. (13.43), the incident power density $\mathcal{P}_i$ at the target T is

$$\mathcal{P}_i = \mathcal{P}_{\text{ave}} = \frac{G_d}{4\pi r^2} P_{\text{rad}} \tag{13.78}$$

The power received at transreceiver O is

$$P_r = A_{er} \mathcal{P}_s$$

or

$$\mathcal{P}_s = \frac{P_r}{A_{er}} \tag{13.79}$$

Note that $\mathcal{P}_i$ and $\mathcal{P}_s$ are the time-average power densities in watts/m^2 and P_{rad} and P_r are the total time-average powers in watts. Since $G_{dr} = G_{dt} = G_d$ and $A_{er} = A_{et} = A_e$, substituting eqs. (13.78) and (13.79) into eq. (13.77) gives

$$\sigma = (4\pi r^2)^2 \frac{P_r}{P_{\text{rad}}} \frac{1}{A_e G_d} \tag{13.80a}$$

or

$$P_r = \frac{A_e \sigma G_d P_{\text{rad}}}{(4\pi r^2)^2} \tag{13.80b}$$

TABLE 13.1 Designations of Radar Frequencies

Designation	Frequency
UHF	300–1000 MHz
L	1000–2000 MHz
S	2000–4000 MHz
C	4000–8000 MHz
X	8000–12,500 MHz
Ku	12.5–18 GHz
K	18–26.5 GHz
Millimeter	>35 GHz

From eq. (13.73), $A_e = \lambda^2 G_d/4\pi$. Hence,

$$P_r = \frac{(\lambda G_d)^2 \sigma P_{\text{rad}}}{(4\pi)^3 r^4} \tag{13.81}$$

This is the *radar transmission equation* for free space. It is the basis for measurement of scattering cross section of a target. Solving for r in eq. (13.81) results in

$$r = \left[\frac{\lambda^2 G_d^2 \sigma}{(4\pi)^3} \cdot \frac{P_{\text{rad}}}{P_r}\right]^{1/4} \tag{13.82}$$

Equation (13.82) is called the *radar range equation.* Given the minimum detectable power of the receiver, the equation determines the maximum range for a radar. It is also useful for obtaining engineering information concerning the effects of the various parameters on the performance of a radar system.

The radar considered so far is the *monostatic* type because of the predominance of this type of radar in practical applications. A *bistatic radar* is one in which the transmitter and receiver are separated. If the transmitting and receiving antennas are at distances r_1 and r_2 from the target and $G_{dr} \neq G_{dt}$, eq. (13.81) for bistatic radar becomes

$$P_r = \frac{G_{dt} G_{dr}}{4\pi}\left[\frac{\lambda}{4\pi r_1 r_2}\right]^2 \sigma P_{\text{rad}} \tag{13.83}$$

Radar transmission frequencies range from 25 to 70,000 MHz. Table 13.1 shows radar frequencies and their designations as commonly used by radar engineers.

EXAMPLE 13.10

An S-band radar transmitting at 3 GHz radiates 200 kW. Determine the signal power density at ranges 100 and 400 nautical miles if the effective area of the radar antenna is 9 m^2. With a 20-m^2 target at 300 nautical miles, calculate the power of the reflected signal at the radar.

Solution:

The nautical mile is a common unit in radar communications.

$$1 \text{ nautical mile (nm)} = 1852 \text{ m}$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{3 \times 10^9} = 0.1 \text{ m}$$

$$G_{dt} = \frac{4\pi}{\lambda^2} A_{et} = \frac{4\pi}{(0.1)^2} 9 = 3600\pi$$

For $r = 100 \text{ nm} = 1.852 \times 10^5 \text{ m}$

$$\mathcal{P} = \frac{G_{dt} P_{\text{rad}}}{4\pi r^2} = \frac{3600\pi \times 200 \times 10^3}{4\pi (1.852)^2 \times 10^{10}}$$
$$= 5.248 \text{ mW/m}^2$$

For $r = 400 \text{ nm} = 4\,(1.852 \times 10^5) \text{ m}$

$$\mathcal{P} = \frac{5.248}{(4)^2} = 0.328 \text{ mW/m}^2$$

Using eq. (13.80b)

$$P_r = \frac{A_e \sigma\, G_d\, P_{\text{rad}}}{[4\pi r^2]^2}$$

where $r = 300 \text{ nm} = 5.556 \times 10^5 \text{ m}$

$$P_r = \frac{9 \times 20 \times 3600\pi \times 200 \times 10^3}{[4\pi \times 5.556^2]^2 \times 10^{20}} = 2.706 \times 10^{-14} \text{ W}$$

The same result can be obtained using eq. (13.81).

PRACTICE EXERCISE 13.10

A C-band radar with an antenna 1.8 m in radius transmits 60 kW at a frequency of 6000 MHz. If the minimum detectable power is 0.26 mW, for a target cross section of 5 m^2, calculate the maximum range in nautical miles and the signal power density at half this range. Assume unity efficiency and that the effective area of the antenna is 70% of the actual area.

Answer: 0.6309 nm, 500.90 W/m^2.

SUMMARY

1. We have discussed the fundamental ideas and definitions in antenna theory. The basic types of antenna considered include the Hertzian (or differential length) dipole, the half-wave dipole, the quarter-wave monopole, and the small loop.
2. Theoretically, if we know the current distribution on an antenna, we can find the retarded magnetic vector potential **A**, and from it we can find the retarded electromagnetic fields **H** and **E** using

$$\mathbf{H} = \nabla \times \frac{\mathbf{A}}{\mu}, \qquad \mathbf{E} = \eta\, \mathbf{H} \times \mathbf{a}_k$$

The far-zone fields are obtained by retaining only $1/r$ terms.
3. The analysis of the Hertzian dipole serves as a stepping stone for other antennas. The radiation resistance of the dipole is very small. This limits the practical usefulness of the Hertzian dipole.
4. The half-wave dipole has a length equal to $\lambda/2$. It is more popular and of more practical use than the Hertzian dipole. Its input impedance is $73 + j42.5\ \Omega$.
5. The quarter-wave monopole is essentially half a half-wave dipole placed on a conducting plane.
6. The radiation patterns commonly used are the field intensity, power intensity, and radiation intensity patterns. The field pattern is usually a plot of $|E_s|$ or its normalized form $f(\theta)$. The power pattern is the plot of $\mathcal{P}_{\text{ave}}$ or its normalized form $f^2(\theta)$.
7. The directive gain is the ratio of $U(\theta, \phi)$ to its average value. The directivity is the maximum value of the directive gain.
8. An antenna array is a group of radiating elements arranged so as to produce some particular radiation characteristics. Its radiation pattern is obtained by multiplying the unit pattern (due to a single element in the group) with the group pattern, which is the plot of the normalized array factor. For an N-element linear uniform array,

$$AF = \left| \frac{\sin(N\psi/2)}{\sin(\psi/2)} \right|$$

where $\psi = \beta d \cos\theta + \alpha$, $\beta = 2\pi/\lambda$, d = spacing between the elements, and α = interelement phase shift.
9. The Friis transmission formula characterizes the coupling between two antennas in terms of their directive gains, separation distance, and frequency of operation.
10. For a bistatic radar (one in which the transmitting and receiving antennas are separated), the power received is given by

$$P_r = \frac{G_{dt} G_{dr}}{4\pi} \left[\frac{\lambda}{4\pi r_1 r_2} \right]^2 \sigma\, P_{\text{rad}}$$

For a monostatic radar, $r_1 = r_2 = r$ and $G_{dt} = G_{dr}$.

REVIEW QUESTIONS

13.1 An antenna located in a city is a source of radio waves. How much time does it take the wave to reach a town 12,000 km away from the city?

(a) 36 s
(b) 20 μs
(c) 20 ms
(d) 40 ms
(e) None of the above

13.2 In eq. (13.34), which term is the radiation term?

(a) $1/r$ term
(b) $1/r^2$ term
(c) $1/r^3$ term
(d) All of the above

13.3 A very small thin wire of length $\lambda/100$ has a radiation resistance of

(a) $\simeq 0\ \Omega$
(b) $0.08\ \Omega$
(c) $7.9\ \Omega$
(d) $790\ \Omega$

13.4 A quarter-wave monopole antenna operating in air at frequency 1 MHz must have an overall length of

(a) $\ell \gg \lambda$
(b) 300 m
(c) 150 m
(d) 75 m
(e) $\ell \ll \lambda$

13.5 If a small single-turn loop antenna has a radiation resistance of $0.04\ \Omega$, how many turns are needed to produce a radiation resistance of $1\ \Omega$?

(a) 150
(b) 125
(c) 50
(d) 25
(e) 5

13.6 At a distance of 8 km from a differential antenna, the field strength is 12 μV/m. The field strength at a location 20 km from the antenna is

(a) 75 μV/m
(b) 30 μV/m
(c) 4.8 μV/m
(d) 1.92 μV/m

13.7 An antenna has $U_{\text{max}} = 10$ W/sr, $U_{\text{ave}} = 4.5$ W/sr, and $\eta_r = 95\%$. The input power to the antenna is

(a) 2.222 W
(b) 12.11 W
(c) 55.55 W
(d) 59.52 W

13.8 A receiving antenna in an airport has a maximum dimension of 3 m and operates at 100 MHz. An aircraft approaching the airport is 1/2 km from the antenna. The aircraft is in the far field region of the antenna.

(a) True
(b) False

13.9 A receiving antenna is located 100 m away from the transmitting antenna. If the effective area of the receiving antenna is 500 cm^2 and the power density at the receiving location is 2 mW/m^2, the total power received is:

(a) 10 nW
(b) 100 nW
(c) 1 μW
(d) 10 μW
(e) 100 μW

13.10 Let R be the maximum range of a monostatic radar. If a target with radar cross section of 5 m^2 exists at $R/2$, what should be the target cross section at $3R/2$ to result in an equal signal strength at the radar?

(a) 0.0617 m^2
(b) 0.555 m^2
(c) 15 m^2
(d) 45 m^2
(e) 405 m^2

Answers: 13.1d, 13.2a, 13.3b, 13.4d, 13.5e, 13.6c, 13.7d, 13.8a, 13.9e, 13.10e.

PROBLEMS

13.1 The magnetic vector potential at point $P(r, \theta, \phi)$ due to a small antenna located at the origin is given by

$$\mathbf{A}_s = \frac{50\, e^{-j\beta r}}{r}\, \mathbf{a}_x$$

where $r^2 = x^2 + y^2 + z^2$. Find $\mathbf{E}(r, \theta, \phi, t)$ and $\mathbf{H}(r, \theta, \phi, t)$ at the far field.

13.2 A Hertzian dipole at the origin in free space has $d\ell = 20$ cm and $I = 10 \cos 2\pi 10^7 t$ A, find $|E_{\theta s}|$ at the distant point (100, 0, 0).

13.3 A 2-A source operating at 300 MHz feeds a Hertzian dipole of length 5 mm situated at the origin. Find $\mathbf{E}_s$ and $\mathbf{H}_s$ at (10, 30°, 90°).

13.4 (a) Instead of a constant current distribution assumed for the short dipole of Section 13.2, assume a triangular current distribution $I_s = I_o\left(1 - \frac{2|z|}{\ell}\right)$ shown in Figure 13.23. Show that

$$R_{\text{rad}} = 20\, \pi^2 \left[\frac{\ell}{\lambda}\right]^2$$

which is one-fourth of that in eq. (13.13). Thus R_{rad} depends on the current distribution.

(b) Calculate the length of the dipole that will result in a radiation resistance of 0.5 Ω.

13.5 An antenna can be modeled as an electric dipole of length 5 m at 3 MHz. Find the radiation resistance of the antenna assuming a uniform current over its length.

13.6 A half-wave dipole fed by a 50-Ω transmission line, calculate the reflection coefficient and the standing wave ratio.

13.7 A 1-m-long car radio antenna operates in the AM frequency of 1.5 MHz. How much current is required to transmit 4 W of power?

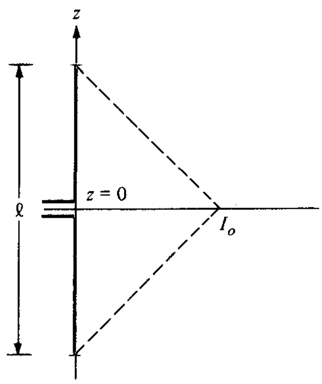

Figure 13.23 Short dipole antenna with triangular current distribution; for Problem 13.4.

***13.8** (a) Show that the generated far field expressions for a thin dipole of length ℓ carrying sinusoidal current $I_o \cos \beta z$ are

$$H_{\phi s} = \frac{jI_o e^{-\beta r}}{2\pi r} \frac{\cos\left(\frac{\beta\ell}{2}\cos\theta\right) - \cos\frac{\beta\ell}{2}}{\sin\theta}, \qquad E_{\theta s} = \eta H_{\phi s}$$

[*Hint:* Use Figure 13.4 and start with eq. (13.14).]

(b) On a polar coordinate sheet, plot $f(\theta)$ in part (a) for $\ell = \lambda$, $3\lambda/2$ and 2λ.

***13.9** For Problem 13.4.

(a) Determine $\mathbf{E}_s$ and $\mathbf{H}_s$ at the far field

(b) Calculate the directivity of the dipole

***13.10** An antenna located on the surface of a flat earth transmits an average power of 200 kW. Assuming that all the power is radiated uniformly over the surface of a hemisphere with the antenna at the center, calculate (a) the time-average Poynting vector at 50 km, and (b) the maximum electric field at that location.

13.11 A 100-turn loop antenna of radius 20 cm operating at 10 MHz in air is to give a 50 mV/m field strength at a distance 3 m from the loop. Determine

(a) The current that must be fed to the antenna

(b) The average power radiated by the antenna

13.12 Sketch the normalized E-field and H-field patterns for

(a) A half-wave dipole

(b) A quarter-wave monopole

13.13 Based on the result of Problem 13.8, plot the vertical field patterns of monopole antennas of lengths $\ell = 3\lambda/2$, λ, $5\lambda/8$. Note that a $5\lambda/8$ monopole is often used in practice.

13.14 In free space, an antenna has a far-zone field given by

$$\mathbf{E}_s = \frac{5 \sin 2\theta}{r} e^{-j\beta r} \mathbf{a}_\theta \text{ V/m}$$

where $\beta = \omega\sqrt{\mu_o \varepsilon_o}$. Determine the radiated power.

13.15 At the far field, the electric field produced by an antenna is

$$\mathbf{E}_s = \frac{10}{r} e^{-j\beta r} \cos\theta \cos\phi \, \mathbf{a}_z$$

Sketch the vertical pattern of the antenna. Your plot should include as many points as possible.

13.16 For an Hertzian dipole, show that the time-average power density is related to the radiation power according to

$$P_{\text{ave}} = \frac{1.5 \sin^2\theta}{4\pi r^2} P_{\text{rad}}$$

13.17 At the far field, an antenna produces

$$\mathbf{P}_{\text{ave}} = \frac{2 \sin\theta \cos\phi}{r^2} \mathbf{a}_r \text{ W/m}^2, \qquad 0 < \theta < \pi, 0 < \phi < \pi/2$$

Calculate the directive gain and the directivity of the antenna.

13.18 From Problem 13.8, show that the normalized field pattern of a full-wave ($\ell = \lambda$) antenna is given by

$$f(\theta) = \frac{\cos(\pi \cos\theta) + 1}{\sin\theta}$$

Sketch the field pattern.

13.19 For a thin dipole $\lambda/16$ long, find: (a) the directive gain, (b) the directivity, (c) the effective area, (d) the radiation resistance.

13.20 Repeat Problem 13.19 for a circular thin loop antenna $\lambda/12$ in diameter.

13.21 A half-wave dipole is made of copper and is of diameter 2.6 mm. Determine the efficiency of the dipole if it operates at 15 MHz.
Hint: Obtain R_ℓ from $R_\ell/R_{\text{dc}} = a/2\delta$; see Section 10.6.

13.22 Find U_{ave}, U_{max}, and D if:

(a) $U(\theta, \phi) = \sin^2 2\theta, \quad 0 < \theta < \pi, 0 < \phi < 2\pi$
(b) $U(\theta, \phi) = 4 \csc^2 2\theta, \quad \pi/3 < \theta < \pi/2, 0 < \phi < \pi$
(c) $U(\theta, \phi) = 2 \sin^2\theta \sin^2\phi, \quad 0 < \theta < \pi, 0 < \phi < \pi$

13.23 For the following radiation intensities, find the directive gain and directivity:

(a) $U(\theta, \phi) = \sin^2\theta, \quad 0 < \theta < \pi, 0 < \phi < 2\pi$
(b) $U(\theta, \phi) = 4 \sin^2\theta \cos^2\phi, \quad 0 < \theta < \pi, 0 < \phi < \pi$
(c) $U(\theta, \phi) = 10 \cos^2\theta \sin^2\phi/2, \quad 0 < \theta < \pi, 0 < \phi < \pi/2$

13.24 In free space, an antenna radiates a field

$$E_{\phi s} = \frac{0.2 \cos^2\theta}{4\pi r} e^{-j\beta r} \text{ kV/m}$$

at far field. Determine: (a) the total radiated power, (b) the directive gain at $\theta = 60°$.

13.25 Derive $\mathbf{E}_s$ at far field due to the two-element array shown in Figure 13.24. Assume that the Hertzian dipole elements are fed in phase with uniform current $I_o \cos\omega t$.

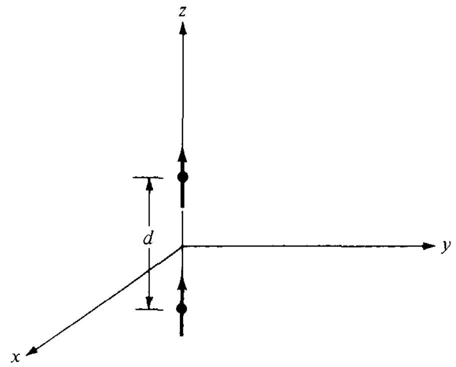

Figure 13.24 Two-element array of Problem 13.25.

13.26 An array comprises two dipoles that are separated by one wavelength. If the dipoles are fed by currents of the same magnitude and phase,

(a) Find the array factor.

(b) Calculate the angles where the nulls of the pattern occur.

(c) Determine the angles where the maxima of the pattern occur.

(d) Sketch the group pattern in the plane containing the elements.

13.27 An array of two elements that are fed by currents that are 180° out of phase with each other. Plot the group pattern if the elements are separated by: (a) $d = \lambda/4$, (b) $d = \lambda/2$

13.28 Sketch the group pattern in the xz-plane of the two-element array of Figure 13.10 with

(a) $d = \lambda$, $\alpha = \pi/2$

(b) $d = \lambda/4$, $\alpha = 3\pi/4$

(c) $d = 3\lambda/4$, $\alpha = 0$

13.29 An antenna array consists of N identical Hertzian dipoles uniformly located along the z-axis and polarized in the z-direction. If the spacing between the dipole is $\lambda/4$, sketch the group pattern when: (a) $N = 2$, (b) $N = 4$.

13.30 Sketch the resultant group patterns for the four-element arrays shown in Figure 13.25.

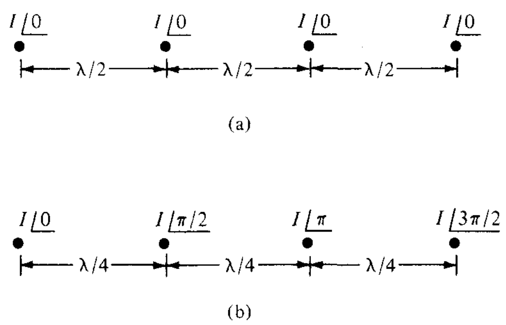

Figure 13.25 For Problem 13.30.

13.31 For a 10-turn loop antenna of radius 15 cm operating at 100 MHz, calculate the effective area at $\theta = 30°$, $\phi = 90°$.

13.32 An antenna receives a power of 2 μW from a radio station. Calculate its effective area if the antenna is located in the far zone of the station where $E = 50$ mV/m.

13.33 (a) Show that the Friis transmission equation can be written as

$$\frac{P_r}{P_t} = \frac{A_{er}A_{et}}{\lambda^2 r^2}$$

(b) Two half-wave dipole antennas are operated at 100 MHz and separated by 1 km. If 80 W is transmitted by one, how much power is received by the other?

13.34 The electric field strength impressed on a half-wave dipole is 3 mV/m at 60 MHz. Calculate the maximum power received by the antenna. Take the directivity of the half-wave dipole as 1.64.

13.35 The power transmitted by a synchronous orbit satellite antenna is 320 W. If the antenna has a gain of 40 dB at 15 GHz, calculate the power received by another antenna with a gain of 32 dB at the range of 24,567 km.

13.36 The directive gain of an antenna is 34 dB. If the antenna radiates 7.5 kW at a distance of 40 km, find the time-average power density at that distance.

13.37 Two identical antennas in an anechoic chamber are separated by 12 m and are oriented for maximum directive gain. At a frequency of 5 GHz, the power received by one is 30 dB down from that transmitted by the other. Calculate the gain of the antennas in dB.

13.38 What is the maximum power that can be received over a distance of 1.5 km in free space with a 1.5-GHz circuit consisting of a transmitting antenna with a gain of 25 dB and a receiving antenna with a gain of 30 dB? The transmitted power is 200 W.

13.39 An L-band pulse radar with a common transmitting and receiving antenna having a directive gain of 3500 operates at 1500 MHz and transmits 200 kW. If the object is 120 km from the radar and its scattering cross section is 8 m^2, find

(a) The magnitude of the incident electric field intensity of the object

(b) The magnitude of the scattered electric field intensity at the radar

(c) The amount of power captured by the object

(d) The power absorbed by the antenna from the scattered wave

13.40 A transmitting antenna with a 600 MHz carrier frequency produces 80 W of power. Find the power received by another antenna at a free space distance of 1 km. Assume both antennas has unity power gain.

13.41 A monostable radar operating at 6 GHz tracks a 0.8 m^2 target at a range of 250 m. If the gain is 40 dB, calculate the minimum transmitted power that will give a return power of 2 μW.

13.42 In the bistatic radar system of Figure 13.26, the ground-based antennas are separated by 4 km and the 2.4 m^2 target is at a height of 3 km. The system operates at 5 GHz. For G_{dt} of 36 dB and G_{dr} of 20 dB, determine the minimum necessary radiated power to obtain a return power of 8×10^{-12} W.

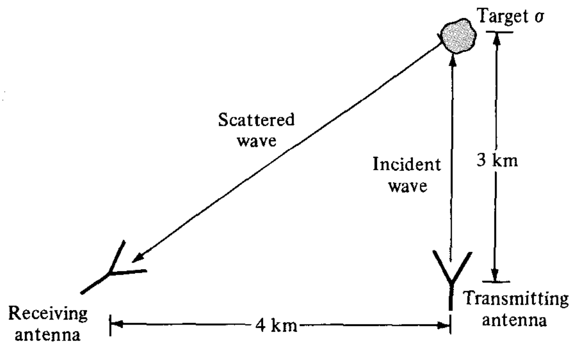

Figure 13.26 For Problem 13.42.